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MODELLING AND STABILITY ANALYSIS IN FUZZY ECONOMICS

R. A. ALIEV †, §

Abstract. The sheer complexity of causation in the economic arena mandates a "fuzzy" approach. We argue that many economic dynamical systems naturally become fuzzy due to the uncertain initial conditions and parameters. In this study the Fuzzy Economics is defined as human-centric and closer to the reality Economics with fuzzy-logic based representation of the economic agent's behavior. We use linguistic rule-base and fuzzy differential equations for modeling the economic agents.

Using fuzzy agents this paper investigates different problems of fuzzy economics and methods of their solutions, namely, time path and stability of the fuzzy dynamic economic systems, fuzzy decision making in oligopolistic economy, neuro-fuzzy time-series forecasting of macroeconomic processes. The suggested approaches to fuzzy macroeconomic optimization and control and prediction problems are successfully applied to path planning and stability analysis of national economy, economic growth, nonlinear manufacture dynamics, crude oil prices forecasting in the world, and portfolio construction. Performance comparisons are made on sound benchmark problems well-known from the literature.

Key words: dynamical systems, fuzzy economics, uncertain initial conditions.

1. Introduction

Macroeconomics changed between the early 1960's and the late 1970's. The macroeconomics of the early 1960's was Keynesian. The decline of the old-style Keynesian economics was results of simultaneous occurrence of increased inflation and unemployment, which was seemed impossible with the simple non-accelerationist Phillips Curves of the early 1960's [1]. But the change was also due to a shift in the world of ideas. In the Keynesian economics its major components, such as consumption function, the investment function, and the price and wage equations were derived from intuition.

These functions were taken from observations as to how the various agents in the economy would behave. The casual ways involved in this methodology are criticized by others suggesting that they should be derived from the behavior of profit maximizing firms and utility-maximizing consumers with objective arguments in their utility functions.

Neoclassical assumptions regarding the opportunity and efficiency of economic agents are not correct. Rational-expectations theory assumes that the economy and the agents within it act with perfect foresight [25].

The advance macroeconomics interprets such behavior through preferences that include norms, which are people's views regarding how they, and others, should or should not behave.

While these preferences are a central feature of sociological theory, but they have been ignored by majority of economists.

Sociologists consider norms to be central to motivation, because people tend to live up to the views and principles they accept and are happy only when they can manage it.

Daniel Kahneman and Amos Tversky [28] have studied people unwillingness to take favorable odd in small bets as due to loss aversion and explain it that people have a mental frame, which makes them reluctant to take losses.

†I Department of Computer-Aided Control System, Azerbaijan State Oil Academy, Azadlig ave. 20, AZ1010 Baku, Azerbaijan; Phone: +99412 493 45 38; Fax: +99412 598 45 09; E-mail: raliev@iatp.az

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Unlike traditional neoclassical theory for behavioural economics assumptions are fundamental to the construction of economic models. The Herbert Simon tradition in behavioural economics maintains that intelligent behaviour need not produce the type of optimal or efficient behaviours predicted by traditional neoclassical theory.

Behavioural economics claims that although people are irrational their irrationality can be modelled with great precision [25].

The behavioural model makes it possible to have multiple equilibria or a fuzzy set solution to the choice set afforded to economic agents in the fundamentally important domain of production. For this reason, the behavioural model can interpret important economic facts that contradict the analytical predictions of neoclassical economic theory which with variety of complex outcomes in competitive equilibrium, which is the substance of economic life.

In a complex environment with a large number of heterogeneous interacting agents there is a high degree of uncertainty about relevant information and their relation. Additionally agents will constantly try to find better representations of the perceived reality and will therefore experience and learn. We need a way of modelling the "mental models" of economic agents that operate within such an economic environment.

Here a powerful tool is fuzzy logic [57], created by prof. L.Zadeh. Using human-centric and closer to reality economics with fuzzy logic based representations of economic agent's behaviour we will call fuzzy economics.

Why Fuzzy Economics?

The sheer complexity of causation in the economic arena mandates a fuzzy approach. The introduction of fuzzy preference relations with more flexible properties makes it possible to describe economic agents with less rational behaviour, who state their choices with greater discrimination.

In most fundamental paradigm of economic relationship, demand-supply analysis, it is assumed that if the demand or supply curve shifts in a certain direction, the price or quantity will change in the direction predicted by the model. Although sometimes we can get the directions of change of the variables, we are rarely able to compute the likely magnitude of change. In this and many other contexts, we are limited to practicing "fuzzy economics" [44].

We get a more complex problem, when both the demand and supply curves shift in the same direction. In this case, we can be confident of the direction of change of quantity, but uncertain as to the direction of change of price. Whether price rises, falls, or remains the same depends entirely upon the relative magnitudes of the shifts of demand and supply. In case when the direction of change is fuzzy, predicting the magnitude of change is irrelevant.

Tracing the economics development since 19th century up to present days makes it evident that at its core is a sequence of rather precise and mathematically sophisticated axiomatic theories. At the same time there is always a noticeable and persistent gap between the economic reality and predictions derived from these theories [31].

Main reasons why economic theories have not been successful in modelling economic reality is directly related with the subject of this paper, i.e. these theories are formulated in terms of classical mathematics, bivalent logic and classical theory of additive measures. Human reasoning and decision making is based on high degrees of uncertainty (usually nonstatistical) and classical mathematics is not capable of expressing this kind of uncertainty. Human preferences for complex choices are not determined, in general, by the rules of additives measures.

Unfortunately, the role of fuzzy logic in economics was recognized much later than in many other areas. British economist Shackle has argued since the late 1940s that probability theory, is not useful for capturing the nature of uncertainly in economics. He has suggested that uncertainty associated with actions with unknown outcomes should be expressed in terms of degrees of possibility rather than by probabilities [49].

A pioneer, who initiated the reformulation of economic theory by using fuzzy logic is C. Ponsard.

First of all C.Ponsard presented the first version of his fundamental article ("An application of fuzzy subsets theory to the analysis of the consumer's spatial preferences") on fuzzy economics, published in Fuzzy Sets and Systems. His thesis was straightforward enough: the new methods of fuzzy optimization make it possible to consider that the objective or the constraint are fuzzy and so account for either the imprecision of the utility function(and so the preferences) or the imprecision of budget constraint.

Then, on this foundation, he developed the symmetrical analysis of the producer [41] after introducing a profit utility function and fuzzifying the technological constraints, then the concept of fuzzy equilibrium of an economy suggested by Arrow-Debreu and finally that of Nash equilibrium.

The concept of a median agent whose behavior allows aggregation of demand correspondences showed that classical hypothesis of the representative agent confronted with imprecise behavior.

Further research results of C. Ponsard included fuzzy preferences, fuzzy utility function, consumer's partial equilibrium, producer's partial equilibrium, fuzzy general equilibria and others.

The paper [13] explores the problem of aggregation of ordinary fuzzy individual preferences into ordinary fuzzy social preferences. The existence of fuzzy models for strict preference, indifference and incomparability satisfying all classical properties is established [38].

Fuzzy individual and social preference relations based new solution concepts in group decision making, and presentation of new soft degrees of consensus are given in [27].

In [14] the expected utility hypothesis as a pattern of rational decision under risk or uncertainty is severely put into question.

In paper [16] it is conducted a comparative review of the application of fuzzy sets theory in economics.

A fruitful scientific centre on Fuzzy Economics is Fuzzy Economics school leaded by Professor Jaime Gil Aluja. The researchers of this school considered Fuzzy microeconomics models, the methods of forecasting under fuzzy uncertainty including Fuzzy Delphi method and other methods [11,12].

Fuzzy microeconomic processes are described in [15]. Reformulated microeconomics is performed via the conception of fuzzy preferences. Since fuzzy preferences introduce a great diversity of possible behaviours of economic agents to the theory of microeconomics, the theory becomes more realistic.

In [24] use of fuzzy set theory and fuzzy logic to construct an annual time-series for the unobservable New Zealand underground economy over the period 1968 to1994 is considered. Two input variables are used – the effective tax rate and an index of the degree of regulation.

In the work [50] it is considered a regulated logistic growth model. The linear stability and the existence of a Hopf bifurcation is investigated. Numerical simulation results are given to support the theoretical predictions.

The book [18] considers simulations of different real world problems, including macro economical problems described by fuzzy differential equations, among which there are supply and demand economic problem, national economy model etc.

In [26] fuzzy control for stabilizing economic processes to implement stabilization strategies in a user-friendly way, by means of a linguistically expressed algorithm is suggested.

Soft Computing based portfolio construction problem is considered in [2]. The proposed model takes into account fuzzy expected return and investor's fuzzy risk preference and gives chance of possibility trade-off between risk and return. This is obtained by assigning degree of satisfaction between criteria and constraints and defining tolerance for the constraints in order to obtain the goal value in the objective risk function. Experimental results demonstrate high efficiency of the proposed method.

In recent years, a great number of papers and some books have explored the use of fuzzy logic as a tool for designing intelligent systems in business, finance, management and economics

[17,21,34,45]. These books present recent progress in the application of constituent Soft Computing (SC) methodologies, in particular neural networks, fuzzy logic, chaos etc. The book [7] highlights some of the recent developments in practical applications of SC in business and economics. It is the first book on application of SC-based hybrid methods combining fuzzy logic, neuro-computing, evolutionary computing, probabilistic computing and chaotic computing in functional areas of business and economics. It brings in a systematic way SC into the university and college educational systems and may be basic text for introducing business managers, teachers, and scientists from various fields of business and economics to the SC technology, enabling them to initiate projects and make applications..

In this paper we consider different problems of fuzzy economics, mainly time path and stability of fuzzy dynamic of economic systems, linguistic modelling of economic agents, neuro-fuzzy time-series forecasting of macroeconomic processes. The paper is structured as follows. In section 2, we cover all prerequisite material (such as fuzzy sets, fuzzy relations, fuzzy functions, fuzzy derivative and fuzzy time series) to be used in the study. In section 3 we present a brief overview of fuzzy modeling of economic agents. In section 4, we discuss the concept of fuzzy stability and necessary criteria of stability of dynamical systems. In Section 5 we formulate the problem of fuzzy decision makingin oligopoly economy. Sections 6, 7, 8 are devoted to stability of various macroeconomic dynamical systems in which fuzzy stability concept plays a vital role. In Section 9 we present fuzzy time series forecasting method based on evolutionary algorithms and neural networks. Section 11 give Soft Computing based portfolio construction. Concluding comments are included in section 11.

2. Preliminaries

In this section, we briefly review some prerequisite material which will be of help in this study. While the reader may find some of the definitions in the literature, we augment them with some interpretation which could be useful in the context of our considerations.

Fuzzy sets [5]. Let X be a classical set of objects, called the universe, whose generic elements are denoted x. Membership in a classical subset A of X is often viewed as a characteristic function μ_A from X to $\{0,1\}$such that

$$\mu_A(x) = \begin{cases} 1 & iff \quad x \in A \\ 0 & iff \quad x \notin A \end{cases},$$

where {0,1} is called a valuation set; 1 indicates membership while 0 - non-membership. If the valuation set is allowed to be in the closed interval [0,1], then A is called a fuzzy set [30,31,34,57,60]. $\mu_A(x)$ is the grade of membership of x in A: $\mu_A : X \to [0,1]$.As closer the value of $\mu_A(x)$ is to 1, so much x belongs to A. A is completely characterized by the set of pairs.

The (crisp) set of elements that belongs to the fuzzy set A at least to the degree α is called the α-level set: $A^\alpha = \{x \in X, \mu_A(x) \geq \alpha\}$. $A^\alpha = \{x \in X, \mu_A(x) > \alpha\}$is called "strong α-level set" or "strong α-cut".

Fuzzy relations. Let $X_1, X_2, ..., X_n$ be nonempty crisp sets. Then $R(X_1, X_2, ..., X_n)$ is called a fuzzy relation of sets$X_1, X_2, ..., X_n$, if $R(X_1, X_2, ..., X_n)$ is the fuzzy subset given on Cartesian product $X_1 \times X_2 \times ... \times X_n$.

Let us consider IF-THEN fuzzy relation**.** Let A and B be fuzzy subsets on the universes of discourse X and Y. To relate the fuzzy subsets A and B of disparate universes of discourse X and Y, the concept of a fuzzy conditional statement (linguistic implication) is introduced, that is

$A \to B$ or "$if\ A\ then\ B$"

Fuzzy function. Briefly speaking, by a fuzzy function we mean a function, whose values are fuzzy numbers (for more details see [5,23,36,39,53]). Let f be a fuzzy function, $\mu_{f(x)}$ denotes the membership function of the fuzzy number$f(x)$, and for $0 < \alpha \leq 1$, $f_r^\alpha(x)$ will denote

$\sup\{z \in dom(\mu_{f(x)}) : \mu_{f(x)}(z) \geq \alpha\}$ and $f_l^{\alpha}(x)$ will denote $\inf\{z \in dom(\mu_{f(x)}) : \mu_{f(x)}(z) \geq \alpha\}$. Functions $f_l^{\alpha}(x)$ and $f_r^{\alpha}(x)$ are level functions of f.

Hukuhara difference [23,32]**.** Let$\tilde{x}, \tilde{y} \in E^n$, where E^n is the space of all fuzzy subsets $\tilde{u}$ of R^n which satisfy the conditions of normality, convexity, and are upper semicontinuous, with compact support [33]. If there exist $\tilde{z} \in E^n$ such that $\tilde{x} = \tilde{y} + \tilde{z}$ then $\tilde{z}$ is called Hukuhara difference of $\tilde{x}$ and $\tilde{y}$ and is denoted as$\tilde{x} -_h \tilde{y}$. Let us recall, that for the standard difference $\tilde{z}$ of $\tilde{x}$ and $\tilde{y}$, $\tilde{x} \neq \tilde{y} + \tilde{z}$. We use Hukuhara difference when we need$\tilde{x} = \tilde{y} + \tilde{z}$.

Fuzzy Hausdorff distance [19]. Let $\tilde{A}$ and $\tilde{B}$ are fuzzy sets, namely$\tilde{A}, \tilde{B} \in E^n$. Then the fuzzy Hausdorff distance $d_{f\mathrm{H}}$ between$\tilde{A}$ and $\tilde{B}$ is defined as

$$d_{f\mathrm{H}}(\tilde{A}, \tilde{B}) = \int_{\alpha} \alpha / d_H(A^{\alpha}, B^{\alpha}), \tag{1}$$

where $d_H(A^{\alpha}, B^{\alpha})$ is the Haussdorff distance between A^{α}, B^{α}.

Let us denote by $\left\| \tilde{A} -_h \tilde{B} \right\| = d_{fH}(\tilde{A} -_h \tilde{B}, \hat{0})$ a fuzzy norm of the Hukuhara difference. We note that $d_{fH}(\tilde{A} -_h \tilde{B}, \hat{0}) = d_{fH}(\tilde{A}, \tilde{B})$. We will be using this difference in further considerations.

Fuzzy norms. Let $\tilde{x}, \tilde{y} \in E^n$. We denote by $\|\tilde{x} -_h \tilde{y}\|_{fH}$ a fuzzy norm defined as

$$\|\tilde{x} -_h \tilde{y}\|_{fH} = d_{fH}(\tilde{x}, \tilde{y}). \tag{2}$$

It is the fuzzy Haussdorf distance mentioned above.

Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n) \in E^n$. We denote by $||\tilde{u}||_f$ a fuzzy norm defined by the formula

$$||\tilde{u}||_f = |\tilde{u}_1| + |\tilde{u}_2| + ... + |\tilde{u}_n|, \tag{3}$$

where $|.|$ is the absolute value of a fuzzy number [32].

Derivatives of fuzzy functions and fuzzy derivatives [29]**.** It is necessary to distinguish between the following cases:

(1) we are given a fuzzy function and our interest is to determine its derivative at a particular point a;
(2) we have a function but the information about the point $\tilde{a}$ at which we are to consider the derivative is vague (uncertain);
(3) we have a fuzzy function and we are interested in its derivative at a vague point $\tilde{a}$.

In this paper, we analyze the situations in which the points are not exactly known, and therefore they need to be substituted by subjective and vague estimates, viz. could be treated as fuzzy sets (numbers) defined over the unit interval.

Possibility measure [5,40]**.** Given two fuzzy sets defined in the same universe of discourse X, a fundamental approach arises as to their similarity or proximity. There are several fundamental approaches introduced in the literature. One of them is a *possibility measure*. The possibility measure, denoted by $Poss(\tilde{A}, \tilde{X})$ describes a level of overlap between two fuzzy sets and is expressed in the form

$$Poss(\tilde{A}, \tilde{X}) = \sup_{x \in X} \left[\tilde{A}(x) t \tilde{X}(x) \right],$$

where t is a t-norm.

Fuzzy differential equations. Let $I = [a, b] \subset R$ and $t_0 > 0$. Assume that fuzzy function $f : I \times \mathrm{E}^n \to \mathrm{E}^n$ is continuous and consider the initial value problem

$$\tilde{x}'(t) = f(t, \tilde{x}(t)), \quad \tilde{x}(t_0) = \tilde{x}_0 \tag{4}$$

A fuzzy valued mapping $\tilde{x} : I \to E^n$ is a solution to the problem (4) if and only if it is continuous and satisfies the integral equation

$$\tilde{x}(t) = \tilde{x}_0 + \int_a^t f(s, \tilde{x}(s))ds \qquad (5)$$

for all $t \in I$.

If $f$is Lipschitz continuous then the problem (4) has a unique solution on$I$. Furthermore, the solution depends continuously on the initial value.

Fuzzy time series. Let's assume, that $U$is the universal set and is given as follows [51]:

$$U = \{u_1, u_2, ..., u_n\}$$

The fuzzy set A of universal set U is defined as follows:

$$A = \mu_A(u_1)/u_1 + \mu_A(u_2)/u_2 + ... + \mu_A(u_n)/u_n$$

Let's assume that $Y(t)$ $(t = 0, 1, 2, 3, ...)$ is a subset of set R^1. In set $Y(t)$ the fuzzy set $\mu_i(t)$ $(t = 1, 2, ...)$are defined and $Y(t)$is universal set for these fuzzy sets. Let's define set $A(t)$, which consists of $\{\mu_i(t)$ $(t = 1, 2, ...)\}$, i.e. $A(t)$ is set of fuzzy sets. Then $A(t)$ is a fuzzy time series in universal set $Y(t)$ $(t = 0, 1, 2, ...)$. If we accept $A(t)$ as linguistic variable, t hen $\mu_i(t)$will be accepted as possible linguistic values of $A(t)$. It can be seen that the function $A(t)$changes with time, and this means that, for different times t**,** function $A(t)$gets different values.

3. Fuzzy modelling of an economic agent

Rule-based models occupy a central position in a broad spectrum of linguistic models [5,40]. Fuzzy-rule-bases can be used for representations of an economic agent's knowledge and usually consist of a set of fuzzy-rules such as:

"If the inflation differential between Europe and the USA is low, then the rate of appreciation of the Euro will be high".

A complete and consistent fuzzy rule-base can be automatically generated on the basis of a numerical database. Nevertheless, the user may influence the results in many ways and therefore always keeps control over the output.

This two criteria, and possible trade-offs have been a subject of many investigations.

The quest for models that are accurate, transparent and user-friendly has characterized intelligent system for decades.

The theory of differential equations has become an essential tool of economic analysis particularly since computer has become commonly available. It would be difficult to comprehend the contemporary literature of economics if one does not understand basic concepts (such as bifurcations and chaos) and results of modern theory of differential equations.

Fuzzy differential equations serve many functions in economics. They are used to determine the conditions for dynamic stability in microeconomic models of market general equilibria and to trace the time path of growth under various conditions and uncertainty in microeconomics. Given the growth rate of a function, differential equations enable the economist to find the function whose growth is described; from point elasticity, they enable the economist to estimate the demand function.

In this study we use linguistic models and fuzzy differential equations for modelling of economic processes.

4. Theory of fuzzy stability

4.1. Fuzzy Stability Concept

Stability is one of the most essential properties of complex dynamical systems, no matter whether technical or human-oriented (social, economical, etc.) In classical terms, the stability property of a dynamical system usually is quantified in a binary form and this quantification states whether the system under consideration reaches equilibrium state after being affected by disturbances. Even if we define a region of stability, in every point of operation of the system we can only conclude that "*the system is stable*" or "*the system is unstable*". No particular quantification as to a degree of stability could be offered. In many cases when such a standard bivalent two-valued definition of stability is being used, we may end up with counterintuitive conclusions.

In contrast, human generated statements would involve degrees of stability which are expressed linguistically and expressed by some fuzzy number which links with some quantification of stability located somewhere in-between absolutely stable and absolutely unstable states, i.e., a stability degree could be expressed by a fuzzy number defined over the unit interval in which 0 is treated as absolutely unstable and 1 corresponds to that state that is absolutely stable. It is then advantageous to introduce linguistic interpretation of degrees of stability, i.e., a degree of stability becomes a linguistic variable assuming terms such as "unstable", "*weakly* stable", "*more or less* stable", "*strongly* stable", "*completely* stable" each of them being described by the corresponding fuzzy numbers defined over [0,1][9].

We can conclude that the concept of stability is a fuzzy concept in the sense that it is a matter of degree.

Accuracy has been a dominant feature in mathematics. However, as the systems under study become more and more complex, nonlinear or uncertain, the use of well-positioned tools of fully deterministic analysis tends to exhibit some limitations and show a lack of rapport with the real world problem under consideration. As a matter of the fact, this form of limitation has been emphasized by the principle of incompatibility [58].

In many cases, information about a behavior of a dynamical system becomes uncertain. In order to obtain a more realistic model, we have to take into account a existing components of uncertainty. Furthermore, uncertainties might not be of probabilistic type. The generalized theory of uncertainty (GTU), outlined by Prof. L.A. Zadeh in [59], breaks with the tradition in real-world problems to view uncertainty as a province of probability and puts it in a much broader perspective. In this setting, the language and formalism of the dynamic fuzzy "if-then" rules and fuzzy differential equations (FDE) become a natural ways to model dynamical systems.

Let us consider a fuzzy differential system

$$\tilde{x'} = f(t, \tilde{x}) \tag{6}$$

where f in (6) is continuous and has continuous partial derivatives $\frac{\partial f}{\partial x}$ on $R_+ \times E^n$, i.e. $f \in C^1\left[R_+ \times E^n, E^n\right]$, and $\tilde{x}(t_0) = \tilde{y}_0 \in E^n$, $t \geq t_0$, $t_0 \in R_+$.

Definition (Zadeh-Aliev). The solution $\tilde{x}(t, t_0, \tilde{y}_0)$ of the system (6) is said to be fuzzy Lipschitz stable with respect to the solution $\tilde{x}(t, t_0, \tilde{x}_0)$ of the system (6) for $t \geq t_0$, where $\tilde{x}(t, t_0, \tilde{x}_0)$ is any solution of the system (6), if there exists a fuzzy number $\tilde{M} = \tilde{M}(t_0) > \tilde{0}$, such that

$$\|\tilde{x}(t, t_0, \tilde{y}_0) -_h \tilde{x}(t, t_0, \tilde{x}_0)\|_{fH} \leq \tilde{M}(t_0)\, \|\tilde{y}_0 -_h \tilde{x}_0\|_{fH} \tag{7}$$

If $\tilde{M}$ independent from t_0, then the solution $\tilde{x}(t, t_0, \tilde{y}_0)$ of the system 6 is said to be uniformly fuzzy Lipschitz stable with respect to the solution $\tilde{x}(t, t_0, \tilde{x}_0)$.

Let $\tilde{x}(t, t_0, \tilde{x}_0)$ be the solution to (6) for $t \geq t_0$. Then $\tilde{\Phi}(t, t_0, \tilde{x}_0) = \frac{\partial \tilde{x}(t,t_0,\tilde{x}_0)}{\partial \tilde{x}_0}$ exists and is the fundamental matrix solution of the variational equation

$$\tilde{z}' = \frac{\partial f(t, \tilde{x}(t, t_0, \tilde{x}_0))}{\partial \tilde{x}} \tilde{z}, \tag{8}$$

and $\frac{\partial \tilde{x}(t,t_0,\tilde{x}_0)}{\partial t_0}$ exists, is a solution of (8), and satisfies the relation:
$\frac{\partial \tilde{x}(t,t_0,\tilde{x}_0)}{\partial t_0} + \tilde{\Phi}(t, t_0, \tilde{x}_0) f(t_0, \tilde{x}_0) = \tilde{0}$, for $t \geq t_0$.

Definition (Zadeh-Aliev-Pedrycz). The solution $\tilde{x}(t, t_0, \tilde{y}_0)$ of the system (6) through $(t_0, \tilde{y}_0)$ for $t \geq t_0$ is said to be ***fuzzy Lipschitz stable*** with respect to the solution $\tilde{x}(t, t_0, \tilde{x}_0)$ of (6) for $t \geq t_0$, where $\tilde{x}(t, t_0, \tilde{x}_0)$ is any solution of the system (6) if and only if there exist $\tilde{M} = \tilde{M}(t_0) > \tilde{0}$ and $\tilde{\delta} > \tilde{0}$ such that $\|\tilde{x}(t, t_0, \tilde{y}_0) - \tilde{x}(t, t_0, \tilde{x}_0)\|_f \leq \tilde{M}(t_0) \|\tilde{y}_0 - \tilde{x}_0\|_f$ for $t \geq t_0$, provided $\|\tilde{y}_0 - \tilde{x}_0\|_f \leq \tilde{\delta}$. If $\tilde{M}$ is independent of t_0, then the solution $\tilde{x}(t, t_0, \tilde{y}_0)$ of the system (6) is ***uniformly fuzzy Lipschitz stable*** with respect to the solution $\tilde{x}(t, t_0, \tilde{x}_0)$.

The theory of FDE which utilizes the Hukuhara derivative has a certain disadvantage that the *diam* $((x(t))^\alpha)$ of the solution $\tilde{x}(t)$ of FDE is a nondecreasing function of time. As it was mentioned in [33], this formulation of FDE cannot reflect any reach behavior of solutions of ODE, such as stability, periodicity, bifurcation and others, is not well suited for modeling purposes. In view of this it is useful to utilize b)-type of strongly generalized differentiability (see Preliminaries). For example, let us consider the following FDE:

$$\tilde{x}' = -\tilde{x}$$

.

The solution of the numeric analog $y' = -y$ of this equation is $y = y_0 e^{-t}$ which is stable. If we use H-derivative (a)-type of strongly generalized differentiability (see Preliminaries)) for the above FDE, then α-cut of its solution is $x_l^\alpha = \frac{1}{2}(x_{l0}^\alpha - x_{r0}^\alpha)e^t + \frac{1}{2}(x_{l0}^\alpha + x_{r0}^\alpha)e^{-t}$, $x_r^\alpha = \frac{1}{2}(x_{r0}^\alpha - x_{l0}^\alpha)e^t + \frac{1}{2}(x_{l0}^\alpha + x_{r0}^\alpha)e^{-t}$. In general it is not stable, and so, we lose stability property. But if we use b)-type of strongly generalized differentiability, then α-cut of its solution is $x_l^\alpha = x_{l0}^\alpha e^{-t}$, $x_r^\alpha = x_{r0}^\alpha e^{-t}$. It is stable and coincides with the solution obtained before for the numeric case.

4.2. Stability Criteria

4.2.1 The Direct method (APG method)

Theorem. Let us consider the solutions $\tilde{x}(t, t_0, \tilde{y}_0)$, $t \geq t_0$, and $\tilde{x}(t, t_0, \tilde{x}_0)$, $t \geq t_0$ of the system (6). Let us assume the following

1) There exists a fuzzy number $\tilde{L}(t_0)$, such that $\int_{t_0}^{\infty} \tilde{\lambda}(s)ds = \tilde{L}(t_0)$, where

$$\tilde{\lambda}(s) \in C\left[[0, \infty), E_+ \subset E\right], E_+ = \left\{\tilde{\lambda} \in E, \operatorname{supp}(\tilde{\lambda}) \geq 0\right\}.$$

2) f satisfies the following fuzzy Lipschitz condition with respect to $\tilde{x}$:

$$\|f(t, \tilde{v}(t, t_0, \tilde{v}_0) + \tilde{x}(t, t_0, \tilde{x}_0)) -_h f(t, \tilde{x}(t, t_0, \tilde{x}_0))\|_{fH} \leq \tilde{\lambda}(t) \|\tilde{v}(t)\|_{fH},$$

where $\tilde{v}(t, t_0, \tilde{v}_0) = \tilde{x}(t, t_0, \tilde{y}_0) -_h \tilde{x}(t, t_0, \tilde{x}_0)$.

Then the solution $\tilde{x}(t, t_0, \tilde{y}_0)$ of the system (6) is fuzzy Lipschitz stable with respect to $\tilde{x}(t, t_0, \tilde{x}_0)$.

4.2.2 The Indirect method (APA method)

Theorem. Let $\tilde{\Phi}(t, t_0)$ be the fundamental matrix of (8). If there exist positive continuous functions $\tilde{k}(t)$ and $\tilde{h}(t) \in E^1$, $t \geq t_0$, such that

$$\int_{t_0}^{t} \tilde{h}(s)||\tilde{\Phi}(t, s)||_f ds \leq \tilde{k}(t) \qquad \texttt{for} \qquad t \geq t_0 \geq 0, \tag{9}$$

and

$$\tilde{k}(t) \exp\left(-\int_{t_1}^{t} \frac{\tilde{h}(s)}{\tilde{k}(s)} ds\right) \leq \tilde{K} \qquad \texttt{for} \qquad t \geq t_1 \geq t_0, \tag{10}$$

where $\tilde{K} \in E^1$ is a fixed positive constant, then the solution $\tilde{x}(t, t_0, \tilde{x}_0)$ of (6) is uniformly fuzzy Lipschitz stable.

5. Fuzzy decision making in oligopoly economy

Here we consider a decision-making system in oligopolistic industry (from USA) [2,10]. In such industry the number of firms is limited to a few, where the actions of one firm have an impact on the industry demand. A number of firms compete for three products (x, y, and z). Each firm is managed by a team, which pursues profitability and market share goals. The teams make marketing, production, and financial decisions each quarter using Decision Support System incorporating data and models. Marketing decisions include the decisions on mix and amounts of goods to produce, pricing, advertising expenses, and other variables. The firm's price and advertising strategies, as well as prices and advertising expenditures of the competitors are the major factors influencing profits, market share, and other marketing variables [46].

The existing decision-making system (DMS) are oriented to econometrics model of the industry history. In particular, as it is shown in [46] existing econometrics models deal inadequately with the information about competitors future behaviour. The models assume the competitor actions are known when the firm makes its decisions. However, the information about competitor future actions is at best vague. Paper [46] presents procedure to model the uncertainties of competitors' behaviour as fuzzy information.

The proposed multi-agent distributed intelligent DMS consists of 5 knowledge based agents, each of which has 9 fuzzy rules. Input information is fuzzy variables of average price $\tilde{x}_1$ (AvgPrice) and average advertising $\tilde{x}_2$ (AvgAdv) of competitors and are the same for all 5 agents. Using fuzzy inference rule each agent produces its output solutions: firms own price $(\tilde{u}_{i1}) i = \overline{1,5}$, and firm's own advertising $\tilde{u}_{i2}, i = \overline{1,5}$. Fuzzy rules in knowledge base of each agent are of the following type (for example, for the first agent):

IF Avg Price is HIGH and AvgAdv is LOW, THEN
Price is HIGH and Advertising is MEDIUM
IF Avg Price is LOW and AvgAdv is HIGH , THEN
Price is MEDIUM and Advertising is HIGH
IF AvgPrice is MEDIUM and Avg Adv is MEDIUM, THEN
Price is HIGH and Advertising is MEDIUM

$\vdots$

In this example, the firms' management must decide price and advertising levels for each quarter of operations. For the first quarter solutions of each agent for situation where AvgPrice is about three hundred and twenty five dollars ($325) and AvgAdv is about sixty thousand ($60,000) are shown in Table 1.

Table 1. Solutions proposed by five agents

Agent #	Price	Advertising
Agent 1	$331.59	$80,000
Agent 2	$331.67	$53,000
Agent 3	$313.61	$60,000
Agent 4	$331.59	$53,000
Agent 5	$331.59	$70,000

The agents in the considered multi-agent distributed intelligent DMS are characterized by the following criteria:

C_1- conformity to situations;
C_2- confidence factor (CF);
C_3- track record.

Fuzzy sets characterizing alternatives on criteria C_i have the following form:

$$\begin{array}{l} C_1 = \{0.7/u_{Ag_1},\ 0.8/u_{Ag_2},\ 0.6/u_{Ag_3},\ 0.7/u_{Ag_4},\ 0.4/u_{Ag_5}\} \\ C_2 = \{0.8/u_{Ag_1},\ 0.7/u_{Ag_2},\ 0.5/u_{Ag_3},\ 0.6/u_{Ag_4},\ 0.3/u_{Ag_5}\} \\ C_3 = \{0.8/u_{Ag_1},\ 0.6/u_{Ag_2},\ 0.4/u_{Ag_3},\ 0.4/u_{Ag_4},\ 0.25/u_{Ag_5}\} \end{array} \quad (11)$$

The optimal alternative is u_{Ag_2}. So Ag_2 is the "winner" agent with the proposal:
Price= \$331.67 ;
Advertising = \$53,000
as the solution of the system.

The final solution of the problem (Price and Advertising for quarters 1, 2, 3, and 4) consists of the sequence of solutions of agent 2 for quarter 1, agent 5 for quarter 2, agent 5 for quarter 3, and agent 3 for quarter 4. All five agents competed for the total solution of the problem. In each quarter the most suitable agent for the situation was the "winner", while the other four agents were losers. For example, for situation in quarter 2 agent 5 was more suitable than the rest of the agents.

6. Stability and time path planning of national economy

A simple model for the national economy of some country may be described by system of fuzzy differential equations

$$\tilde{I}' = \tilde{I} - \tilde{a}\tilde{C}, \quad \tilde{a} > 1 \quad (12)$$

$$\tilde{C} = \tilde{b}(\tilde{I} - \tilde{C} - \tilde{G}), \quad \tilde{b} \geq 1 \quad (13)$$

$$\tilde{G} = \tilde{G}_0 + \tilde{k}\tilde{I}$$

where $\tilde{I}$ - the national income, $\tilde{C}$ - the rate of consumer spending, $\tilde{G}$ - the rate of government expenditures and $\tilde{I}'(\tilde{C}')$ is the time derivative of $I(C)$.

All the parameters are $\tilde{a}$, $\tilde{b}$, $\tilde{G}_0$, $\tilde{k}$ and the two initial conditions. Let us assume that the government effect is fuzzy and the other parameters are all unknown and fuzzy.

The solutions $\tilde{I}(t)$ and $\tilde{C}(t)$ are also fuzzy. The system now becomes a continuous fuzzy system whose trajectories are fuzzy so that any slice through a trajectory at some time $t$is a fuzzy number. We wish to estimate the stability and time path growth of given economical system.

Let $k = 0$, and the other parameters are described by triangular fuzzy numbers: $\tilde{G}_0 = (20000/30000/40000)$, $\tilde{a} = (1.4/1.5/1.6)$, $\tilde{b} = (1.1/1.2/1.3)$. Let the initial conditions be also described by triangular fuzzy numbers:

$$\tilde{I}_0 = (90000/100000/110000), \tilde{C}_0 = (75000/80000/85000)$$

In this case the system is fuzzy Lipschitz stable because $||\tilde{\Phi}||_f \leq M$, where $\tilde{\Phi}$ is the fundamental matrix solution of the system (12)-(13). It means that the inequality

$$\|\tilde{x}(t, t_0, \tilde{y}_0) - \tilde{x}(t, t_0, \tilde{x}_0)\| \leq \tilde{M}(t_0)\,\|\tilde{y}_0 - \tilde{x}_0\|\,,$$

is satisfied with $\tilde{M}$ being a triangular fuzzy number:

$$\tilde{M} = (2.668/2.9526/3.4165).$$

The fuzzy degree of stability is

$$\widetilde{Deg} = (0.9274/0.99/1).$$

The degree was calculated by the following formula:

$$\widetilde{Deg} = \frac{\int_0^{\tilde{\delta}} \left(\tilde{M} ||\Delta \tilde{x}_0|| - ||\Delta \tilde{x}|| \right) d \, \|\Delta \tilde{x}_0\|}{\int_0^{\tilde{\delta}} \left(\tilde{M} ||\Delta \tilde{x}_0|| \right) d \, \|\Delta \tilde{x}_0\|} \tag{14}$$

The graphs of the National Income and Consumer Spending which illustrate the behavior of the stable system are shown below in the Figures 1 and 2

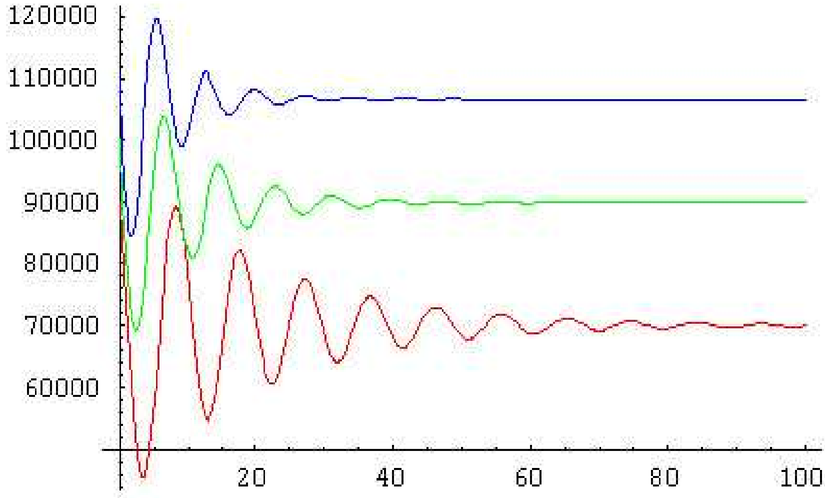

Figure 1. Fuzzy Trajectory for National Income (Core and support behavior)

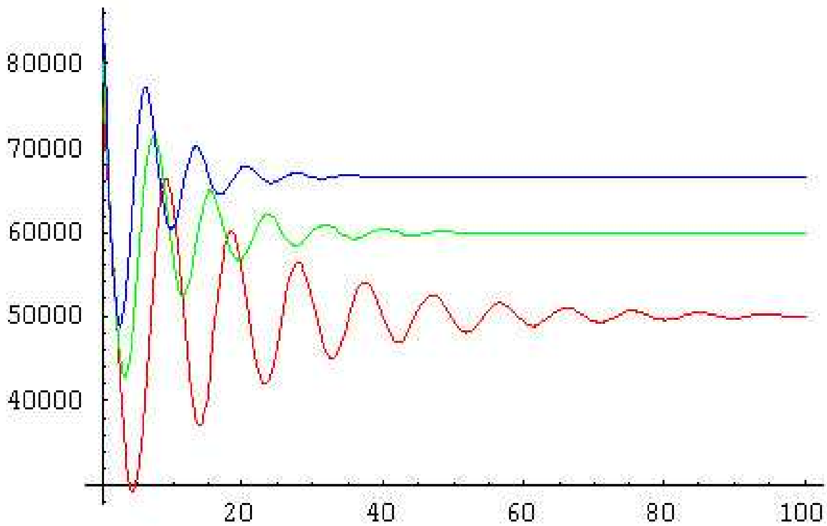

Figure 2. Fuzzy Fuzzy Trajectory for Consumer Spending (Core and support behavior)

Let us now change only the parameter b:

$$k = 0, \tilde{G}_0 = (20000/30000/40000), \tilde{a} = (1.4/1.5/1.6), \tilde{b} = (0.7/0.8/0.9),$$

$$\tilde{I}_0 = (90000/100000/110000), \tilde{C}_0 = (75000/80000/85000).$$

In this case the economic system is unstable and one can see it looking at Figures 3 and 4

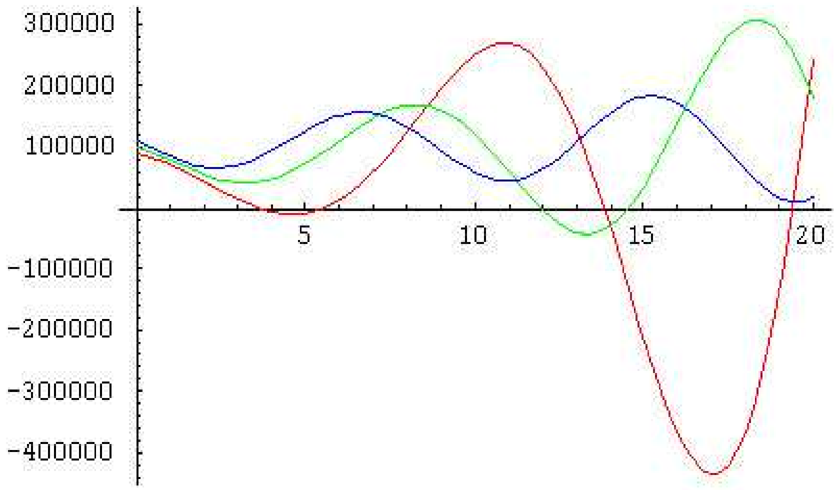

FIGURE 3. Fuzzy Trajectory for National Income (Core and support behavior)

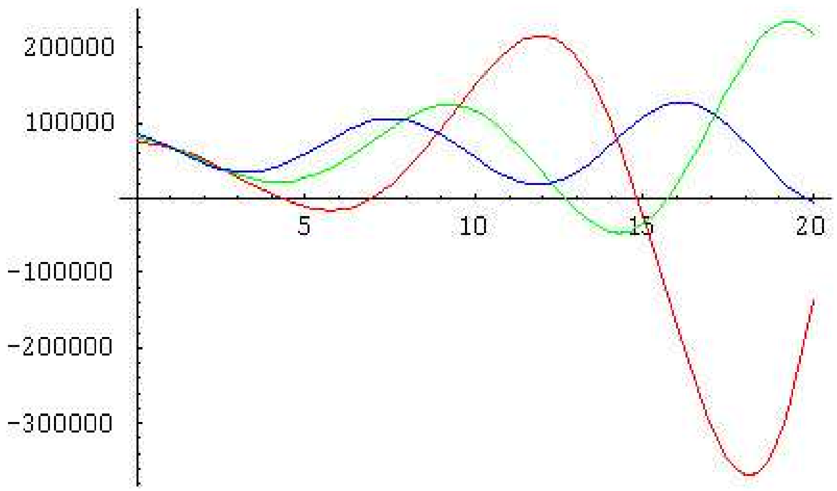

FIGURE 4. Fuzzy Trajectory for Consumer Spending (Core and support behavior)

7. Economic growth control problem

Let us consider the problem of control of macroeconomic system's behavior in long-term period. We assume that the system is competitive and macroeconomic trajectories are comes out from behavior of a large amount o interacting producers and consumers (agents). The model of the economical growth is described by a fuzzy differential equation:

$$\tilde{k}' = \tilde{s}\tilde{k}^{0.5} - \tilde{v}\tilde{k} - \tilde{c} \qquad (15)$$

Here $\tilde{k} = \tilde{K}/\tilde{L}$, $\tilde{k}$ - a capital ratio; $\tilde{K}$ - capital; $\tilde{L}$ - labor; $\tilde{s}$ - proportionality constant between rate of change of capital and production output of economy, $\tilde{\nu}$ is the per capita growth rate, $\tilde{c}$ is the consumption per labor. We do not assume that gross investment is a fixed part of a product. On the contrary, in control model we look for in some sense the best correlation between consumption and investment. It is needed the choice of an optimal value of non-productive consumption which defines size of investment, size of a capital and, therefore, size of an output.

In our problem the state variable is the capital ratio, and the control variable that defines a trajectory of economic growth in a perspective is the consumption function per effective labor. It is needed to synthesize an economic system with a given degree of stability.

Choosing $\tilde{c}$ in the form $\tilde{c} = \tilde{b}\tilde{k}$ we obtain different degrees of stability for different values of $\tilde{b}$. Let the parameters of the system (15) and initial conditions are described by triangular fuzzy numbers:

$$\tilde{s} = (0.297/0.33/0.363), \tilde{v} = (0.036/0.04/0.044), \tilde{k}_0 = (34/35/36).$$

For the case when $\tilde{b} = 0.01$ the system is stable and the fuzzy degree of stability is a triangular fuzzy number

$$\widetilde{Deg} = (0.568/0.635/0.635).$$

Thus we can say that the system is "*more or less*" stable. The degree was calculated by the formula (15).

The graph of the capital ratio dynamics which illustrates the behavior of the stable system is shown below in the Figure 5.

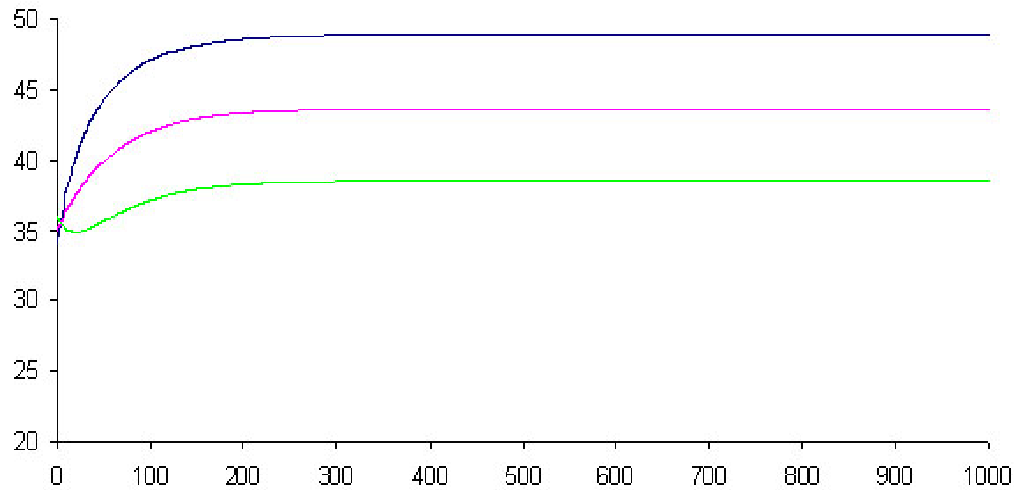

Figure 5. Fuzzy Trajectory of a capital ratio (Core and support behavior)

For the case when $\tilde{b} = 0.02$ the system is stable and the fuzzy degree of stability is a triangular fuzzy number

$$\widetilde{Deg} = (0.76/0.85/0.85)$$

So, we can conclude that the system is "*strongly*" stable.

The graph of the capital ratio dynamics which illustrates the behavior of the stable system is shown belowin the Figure 6.

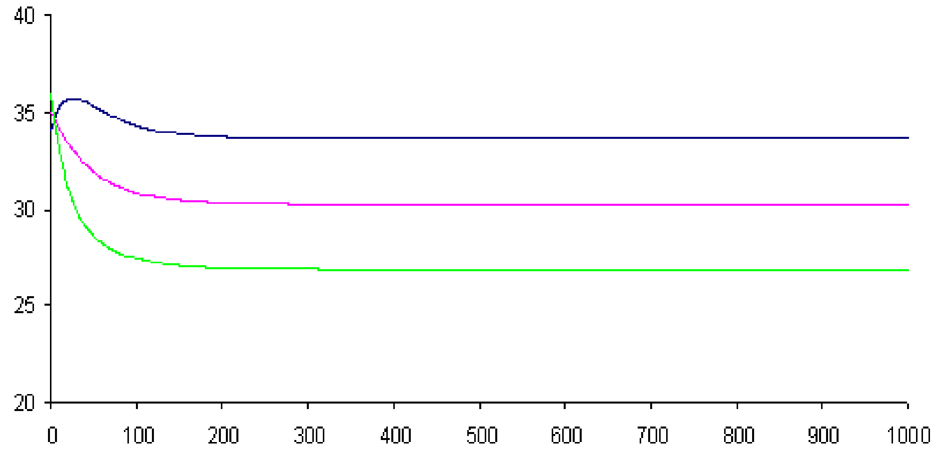

Figure 6. Fuzzy Trajectory of a capital ratio (Core and support behavior)

8. Fuzzy nonlinear model of a manufacture dynamics

A simple nonlinear model describing the dynamics of a manufacture represents the relation between the level of relative production output and its rate of change [54]:

$$\frac{d\tilde{q}(t)}{dt} = \tilde{q}(t) -_h \tilde{q}^2(t), \tilde{q}\,(0) = \tilde{q}_0, \tilde{q}(t), \tilde{q}_0 \in E^1, \qquad (16)$$

where $\tilde{q}(t)$ is a relative production output. $\tilde{q}(t) = \tilde{Q}/\tilde{Q}^*$, where $\tilde{Q}$ is production output and $\tilde{Q}^*$ is its equilibrium value [54] Production output is one of the main indices of macroeconomics.

For $\alpha = 1$, the fundamental matrix solution is $\Phi^{\alpha=1} = \frac{e^{t+t_0}}{\left(e^{t_0}\left(q_0^{\alpha=1}-1\right)-e^t q_0^{\alpha=1}\right)^2}$.

The graphs of the two fuzzy solutions of (16) with the initial conditions $q_{l0}^{\alpha=0} = 0.3$, $q_0^{\alpha=1} = 0.4$, $q_{r0}^{\alpha=0} = 0.5$, $qq_{l0}^{\alpha=0} = 0.8$, $qq_0^{\alpha=1} = 0.9$, $qq_{r0}^{\alpha=0} = 1$ are shown in the Figure 7.

If $q_{i0}^{\alpha=0} \geq 0.1$ and $t_0 = 0$ then $||\tilde{\Phi}(t, t_0, \tilde{q}_0)||_f \leq \tilde{M}$, where $\tilde{M} = (1, 1, 2.8)$, and thus $\|\tilde{q}(t, t_0, \bar{\tilde{q}}_0) -_h \tilde{q}(t, t_0, \tilde{q}_0)\|_f \leq \tilde{M}\, \|\bar{\tilde{q}}_0 -_h \tilde{q}_0\|_f\, l.$

$\tilde{M}$ is independent of t_0, and thus solution of (16) is uniformly fuzzy Lipschitz stable in variation. The degree of stability is $\widetilde{Deg} = (0.396562, 0.5404, 0.679825)$. The possibility measures of the similarities between this fuzzy number and the terms "*weakly* stable", "*more or less* stable" and "*strongly* stable" are about 0.26, 0.918, 0.37 respectively, i.e. the system is more or less stable.

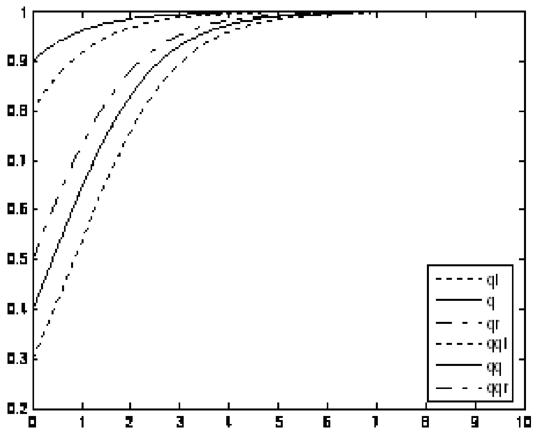

Figure 7. Plot of the fuzzy solutions to (16)

9. Fuzzy time-series forecasting

Assume that an unknown nonlinear system is expressed as follows[32]:

$$\tilde{y}(t) = \tilde{g}\left(\tilde{y}(t-1), ..., \tilde{y}(t-n), u(t-1), ..., u(t-m)\right), \tag{17}$$

where $\tilde{y}(t)$ and $\tilde{u}(t)$ are the output and input of the system, respectively, represented as fuzzy valued function, $\tilde{g}(.)$ is an unknown nonlinear fuzzy mapping to be estimated by RFNN, n and m are order of the system. It is required to design RFNN such that its output $\tilde{y}_N(t)$ determined as

$$\tilde{y}_N(t) = \tilde{g}_N\left(\tilde{y}(t-1), \tilde{u}(t-1), \tilde{W}, \tilde{V}, \tilde{\theta}\right) \tag{18}$$

will be as close as possible to $\tilde{y}(t)$ (17), where $\tilde{W}, \tilde{V}, \tilde{\theta}$ collectively define the structure and set of parameters of RFNN: forward connection weights, backward (recurrent) connection weights, and biases, respectively.

As a measure of closeness between $\tilde{y}(t)$ and $\tilde{y}_N(t)$ we need to define a suitable error function serving as a distance (metric). For continuous variables there is a long list of distance functions [32,36]. In this paper we will use the well-known and commonly used Hamming distance. Therefore the problem of learning of RFNN is an optimization problem with the purpose of adjusting fuzzy parameters $\tilde{W} = \{\tilde{w}_{lij}\}$, $\tilde{V} = \{\tilde{v}_{lij}\}$, and $\tilde{\theta} = \{\tilde{\theta}_{li}\}$ to minimize the error function [57]

$$\tilde{E} = \sum\sum |\tilde{y}_{pi} - \tilde{y}_{Npi}| \tag{19}$$

where $\tilde{y}_{pi}$ is the desired value and $\tilde{y}_{Npi}$ is the actual value of RFNN output layer's neuron i when applied training patter p.

As optimization strategy for training RFNN we will use an evolutionary computing strategy, namely, Differential Evolution Optimization (DEO) method.

During the training, the weights of feed-forward and feed-back connections and biases of RFNN are optimized by the DE algorithm which would lead to the minimum of error function (19).

We start with non-linear system studied in [20,33,35,47] as a benchmark identification problem.

The dynamic system is described by the equation:

$$\tilde{y}(k) = g(\tilde{y}(k-1),\, \tilde{y}(k-2)) + \tilde{u}(k) \qquad (20)$$

where

$$g(\tilde{y}(k-1), \tilde{y}(k-2)) = \frac{\tilde{y}(k-1)\tilde{y}(k-2)(\tilde{y}(k-1)-0.5)}{1+\tilde{y}^2(k-1)+\tilde{y}^2(k-2)} \qquad (21)$$

The system output depends on both its past values and current input. The goal is to approximate the model (20)-(21) by RFNN.

The RFNN for this example has 2 input neurons, 6 neurons at layer 1 and one output neuron. The number of all connections (including forward, backward, and biases) was 62. On the basis of (21) 400 data were created using random (in interval [-1,1] signal u and used for training. The trained network was tested on the basis of 200 test data created using (21) by applying sinusoidal signal $u = \sin(2\pi k/25)$.

DEO progress (MSE vs. successful iterations) is shown in Figure 8.

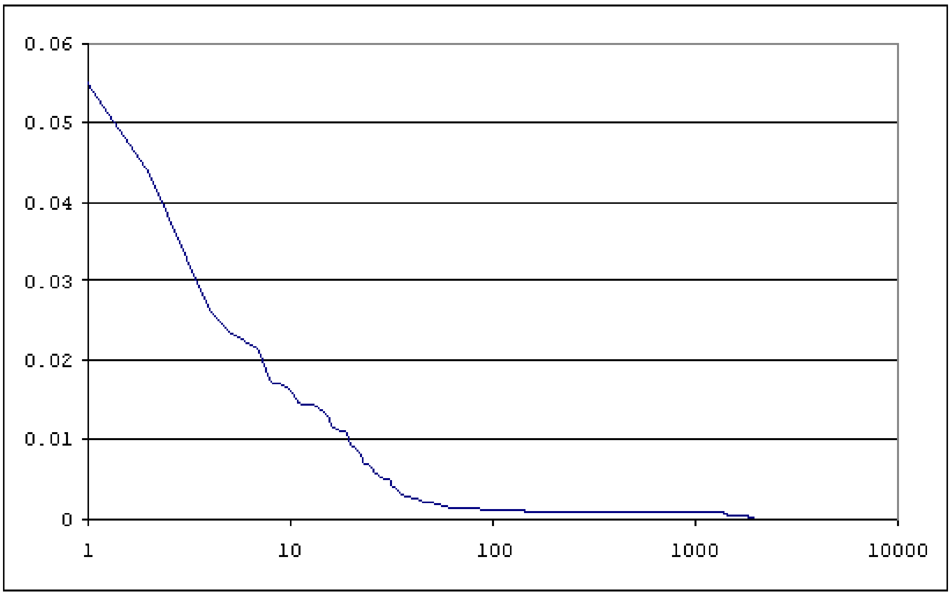

Figure 8. RFNN error convergence

The final reached MSE at the best experiment was 0.000024 on training data and 0.000066 on test data. Table 2 presents the comparative results obtained by different methods given in literature.

Table 2. Comparative results by different methods

Reference fuzzy model	MSE on training set	MSE on test set
[47]	-	0.00080
[46]	0.00075	0.00035
[43]	0.00010	0.00032
RFNN (our approach)	0.000024	0.000066

The RFNN parameters are listed in Table 3 ($\tilde{w}_{lij}$ means weight of the forward link from neuron j at layer $(l-1)$ to neuron i at layer l, $\tilde{v}_{lij}$ means weight of the backward link from neuron j to neuron i both at layer l, t_{li} means bias (threshold) of neuron i at the layer l).

Table 3. Parameters of atrained RFNN

RFNN parameters	Fuzzy triangle value
$\tilde{w}_{l11}$	[-0.4314, -0.1094, 0.8968]
$\tilde{w}_{l12}$	[-0.6209, -0.2578, 0.2284]
t_{l1}	[-0.3902, -0.1195, 0.4474]
$\tilde{w}_{l21}$	[-0.2801, -0.2204, -0.0352]
$\tilde{w}_{l22}$	[-0.3829, -0.2128, 0.5998]

Sun-spot prediction

The performance of RFNN was also tested on a well-known problem of sun-spot prediction [48,49]. Sunspot numbers rise and fall with an irregular cycle with a length of approximately 11 years. In addition to this, there are variations over longer periods. The recent trend is upward from 1900 to the 1960s, then somewhat downward. The historical data for this problem were taken from the Internet. Several data sets were prepared as in [6,48]. The data used for training were sun-spot data from years 1700 to 1920. Two unknown prediction sets used for testing were from 1921 to 1955 (PR1) and from 1956 to 1979.

The comparison of performance of the RFNN approach with other existing methods for two different datasets (PR1, PR2) is presented in Table 4 (NMSE i.e. the Normalized Mean Square Error measure is used in these experiments).

Table 4. MSE obtained by different models for sun-spot prediction

Author (Method)	**Number of inputs**	**PR1**	**PR2**
Rementeria (AR) [50]	12	0.126	0.36
Tong (TAR) [51]	12	0.099	0.28
Subba Rao (Bilinear) [52]	9	0.079	-
DeGroot (ANN)[53]	4	0.092	-
Nowland (ANN) [54]	12	0.077	-
Rementeria (ANN) [50]	12	0.079	0.34
Waterhouse (HME) [55]	12	0.089	0.27
(RFNN-1)	1	0.066	0.22
(RFNN-2)	1	0.074	0.21

The last two rows in Table 4 were obtained by two networks trained on the same data sets by two different persons independently (indicated RFNN-1 and RFNN-2, respectively). In RFNN-1 and RFNN2 the total numbers of neurons were 9 (1+7+1) and 13 (1+11+1), respectively. The numbers of connections for RFNN-1 and RFNN-2 were 148 and 179, respectively.

Table 4 presents comparative results on performance of different forecasting methods for sun-spot prediction problem.

As can be seen from Table 4, the suggested RFNN has simpler structure (having only 1 input neuron) than other models. The identification error of the RFNN is less than that of existing models applied to sun-spot forecasting problem.

Forecasting of crude oil prices in the world

In this example the problem is to forecast crude oil prices in the world. In our fuzzy forecasting model we assumed the relationship:

$$y(k+1) = F(y(k-1), y(k)),$$

For this example we used actual daily data set from Internet (Table 5). Approximately 70% of the data were used for training and the reaming data were used for testy of RFNN. Experimental results demonstrate that DEO training based RFNN showed superior performance with acceptable MSE. Manly forecasting is performed by same procedures.

Table 5. Oil pricedata

Days	Actual	Forecasting	error
66	83.34	83.8325	0.005875
67	84.46	85.16335	0.008259
.	.	.	.
.	.	.	.
.	.	.	.
90	95.81	95.56985	0.002513
91	95.11	95.10399	6.32E-05
92	98.12	97.01907	0.011348
93	95.86	95.61927	0.002518
94	96.25	95.86884	0.003976
95	94.89	94.95197	0.000653
96	97.88	96.87511	0.010373
97	95.91	95.65212	0.002696
		94.78168	
		MSE=0,7255%	

10. Soft Computing based portfolio construction

The fuzzy portfolio optimization model can be formulated as follows [23]:

$$z = \frac{1}{T}\sum_{t=1}^{T}\left|\min\{0, \sum_{j=1}^{n}(r_{jt} - \tilde{R}_j)\tilde{x}_j\}\right| \to \min \quad (22)$$

$$\begin{array}{l} \sum_{j=1}^{n} \tilde{R}_j \tilde{x}_j \geq \tilde{\rho} - q(1-\mu_B(y)) \ , \\ \sum_{j=1}^{n} \tilde{x}_j = 1 \\ \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n \geq 0 \end{array} \quad (23)$$

Here $\tilde{R}_j$ is the fuzzy value of expected return, $\tilde{\rho}$ is the minimum rate of return, $\tilde{x}_j$ are fuzzy values of portfolio allocation in security j over the entire period T, and r_{ij} is return of security

j over period t. The problem is to determine such values of $\tilde{x}_j$ under fuzzy inequalities and equality conditions (23), that would minimize the fuzzy value of objective function (22).

The input data for fuzzy portfolio selection model are historical returns data and fuzzy values of expected returns in future. The expected values of returns are determined by expert perception by using fuzzy numbers. A fuzzy portfolio selection model based on fuzzy linear programming (22)-(23) was solved by genetic algorithm that provides for finding a global near-optimal solution with a reduction in computational complexity compared to the existing methods.

Using portfolio model (22) and (23) and statistical data, an efficient frontier for portfolio model is constructed. The efficient frontier is obtained for different values of portfolio return. In the simulation fifty chromosomes are generated for each asset. In Figure 9 the portfolio efficient frontier for twelve stocks, after defuzzification of the fuzzy return and fuzzy risk, is shown.

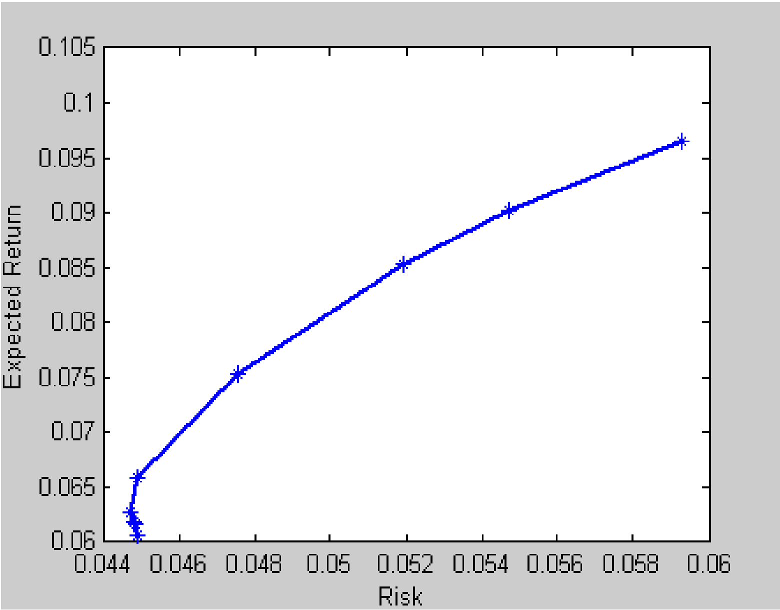

FIGURE 9. Fuzzy portfolio efficient frontier

For comparison, deterministic portfolio selection modelling was carried out using the same historical data. For each stock the expected returns were determined through arithmetic means of historical return rates. Then, applying deterministic semiabsolute deviation model and genetic algorithm the optimal values of invested proportions were determined. In Figure 10 the efficient frontier which is constructed by deterministic model is shown. Note that the values of objective risk function in fuzzy portfolio are less than in deterministic models. It can be said, therefore, that deterministic model is more risky than the fuzzy model.

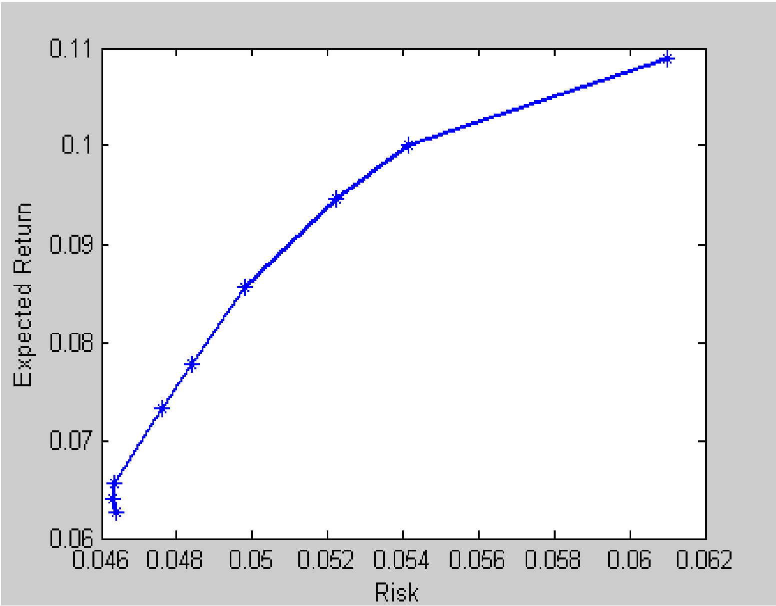

FIGURE 10. Efficient frontier obtained from deterministic model

In this work a fuzzy portfolio selection model based on fully fuzzified linear programming with fuzzy goal (fuzzy risk), fuzzy constraints (fuzzy expected return), and fuzzy variables (fuzzy invested proportions) is suggested. To solve a portfolio selection problem on the basis of this model, we use genetic algorithm which provides a global near-optimal solution without conversion of the initial natural fuzzy linear programming (LP) method to crisp linear programming models. The model provides higher accuracy of solution with less computational complexity in near-optimal portfolio construction.

The use of fully fuzzified LP model with soft constraints and genetic algorithm for solving the portfolio selection problem allows management of the conflict between expected rate of return rate and risk by providing a trade-off between them.

11. Concluding comments

In this study we have discussed a concept of fuzzy economics defined as human-centric and realistic economics with fuzzy based representation of the economic agent's behaviour. We have developed the new approach to investigation of stability and time path planning of macroeconomic systems described by fuzzy differential equations.

Multi-agent distributed intelligent system based on fuzzy decision making for an oligopolistic industry is suggested.

The new effective fuzzy time-series forecasting method on the base of recurrent neural networks and evolutionary algorithms is proposed. Comparison analysis made on sound benchmark problems showed superior performance of this method.

The proposed methods and tools were applied to different macroeconomic problems, namely, to analysis of national economy, economic growth, manufacture dynamics, portfolio optimization and crude oil prices forecasting. The results have demonstrated the efficiency of the suggested approaches.

We would like to stress, however, that the fuzzy economics is in its preliminary stage and further developments are needed.

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Rafik Aliev - was born in Agdam, Azerbaijan, in 1942. He received Ph.D. degree in 1967 and Doctorate in 1975 from Institute of Control Problems (Moscow). He is a Professor and the Head of Department of Computer-Aided Control Systems in Azerbaijan State Oil Academy (Baku, Azerbaijan). He is a visiting professor in University of Siegen (Siegen, Germany) and in Near East University (North Cyprus). He is also an invited speaker in Georgia State University (Atlanta, Georgia, USA). Rafik Aliev is a member of IEEE, Berkeley Initiative in Soft Computing (BISC) Group and International Fuzzy Systems Association (IFSA). He is also a regular Co-Chairman of the International Conferences on Applications of Fuzzy Systems and Soft Computing (ICAFS) and International Conferences on Soft Computing and Computing with Words (ICSCCW).

He is a member of Editorial Boards of International Journal of Web based Communities (Netherlands), Iranian Journal of Fuzzy Systems (Iran), International Journal of Advances in Fuzzy Mathematics (Italy), International Journal of Computers and Electrical Engineering (UK), International Journal "Intelligent Automation and Soft Computing". His research interests cover Modern Control Theory, Soft Computing and Multi-Agent Distributed Artificial Intelligence, Computing with Words. His current research is focused on Generalized Theory of Stability, Recurrent Fuzzy Neural Networks, Fuzzy Type-2 systems, Evolutionary Computation, and Behavioral Economics. Rafik Aliev was awarded USSR State Prize in field of Science (1983) and USA Fulbright Award (1997).