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# NEWTON-TAU NUMERICAL SOLUTION OF A SYSTEM OF NONLINEAR FREDHOLM INTEGRAL EQUATIONS OF SECOND KIND

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ABSTRACT. In this paper, the Newton's method is applied to linearize a system of nonlinear Fredholm integral equations of second kind and obtain a system of linear Fredholm integral equations and so we found the Tau's numerical solution of the linear form.
Key words: System of Nonlinear Integral equations, Newton's method, Tau's method.
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## 1. Introduction

Integral equations are used as mathematical models for many physical situations, and integral equations also occur as reformulations of other mathematical problems. Recently a great deal of interest has been focused on the applications of the Newton's method or Tau's method to solve a wide variety of problems [1,2,3,4,5,6]. The Taylor-series expansion method presented in [7] for solving a system of linear Fredholm integral equations of second kind. In our work the Newton's method and Tau's method applied for solving a system of nonlinear Fredholm integral equations with smooth or weakly singular nonlinear kernels.
Let us consider the following model for a system of nonlinear Fredholm integral equations of second kind

$$U(x) = F(x) + \lambda. \int_a^b K(x, t, u_1(t), u_2(t), \dots, u_d(t))\, dt, \quad x \in [a, b], \tag{1.0.1}$$

where

$$U(x) = [u_1(x), u_2(x), \dots, u_d(x)]^T,$$
$$F(x) = [f_1(x), f_2(x), \dots, f_d(x)]^T,$$
$$K(x, t, u_1(t), \dots, u_d(t)) = [K_1(x, t, u_1(t), \dots, u_d(t)), \dots, K_d(x, t, u_1(t) \dots, u_d(t))]^T,$$
$$\lambda = [\lambda_1, \lambda_2, \dots, \lambda_d]^T,$$

The functions $F$ and $K$, and the vector $\lambda$ are given, and $U$ is the solution to be determined.
For $R = [R_1, \dots, R_d]^T$, we define

$$\lambda.R = [\lambda_1 R_1, \lambda_2 R_2, \dots, \lambda_d R_d]^T.$$

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## 2. Application of the Newton's method

Now we apply the Newton's method to linearization of the problem (1.0.1). For this purpose, we assume that:

(1) $f_i \in C[a,b], \quad i = 1, \ldots, d,$
(2) $K_i \in C([a,b] \times [a,b] \times R^d), \quad i = 1, \ldots, d,$ are continuously differentiable with respect to $u_i$ for $i = 1, \ldots, d$.

We introducing an operator $T : W^d \to W^d$, $W = C[a,b]$, through the formula,

$$T(U)(x) = U(x) - F(x) - \lambda. \int_a^b K(x, t, u_1(t), \ldots, u_d(t))\, dt, \quad x \in [a,b].$$

So, the integral equations (1.0.1) can be written in the form

$$T(U) = 0. \tag{2.0.2}$$

Newton's method for this problem is

$$U_{m+1} = U_m - [T^{'}(U_m)]^{-1} T(U_m), \quad m = 0, 1, 2, \ldots,$$

or equivalently,

$$T^{'}(U_m)(U_{m+1} - U_m) = -T(U_m)$$

where $T^{'}(U_m)$ is the Frechet derivative of T at $U_m$. Let us compute the derivative of $T$,

$$T^{'}(U)(V)(x) = \lim_{h \to 0} \frac{1}{h}[T(U + hV)(x) - T(U)(x)] \tag{2.0.3}$$

$$= \lim_{h \to 0} \frac{1}{h}[hV(x) - \lambda. \int_a^b (K(x, t, u_1(t) + hv_1(t), \ldots, u_d(t) + hv_d(t)) - K(x, t, u_1(t), \ldots, u_d(t)))dt]$$

$$= V(x) - \lambda. \int_a^b \kappa(U(t)) V(t) dt$$

where

$$\kappa(U(t)) = \begin{pmatrix} \frac{\partial K_1(x,t,u_1(t),\ldots,u_d(t))}{\partial u_1} & \cdots & \frac{\partial K_1(x,t,u_1(t),\ldots,u_d(t))}{\partial u_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial K_d(x,t,u_1(t),\ldots,u_d(t))}{\partial u_1} & \cdots & \frac{\partial K_d(x,t,u_1(t),\ldots,u_d(t))}{\partial u_d} \end{pmatrix}$$

and

$$V(x) = [v_1(x), \ldots, v_d(x)]^T.$$

Therefore, the corresponding Newton's iteration formula is

$$\delta_{m+1}(x) - \lambda. \int_a^b \kappa(U_m(t)) \delta_{m+1}(t) dt = -U_m(x) + F(x) + \lambda. \int_a^b K(x, t, u_{m,1}(t), \ldots, u_{m,d}(t))\, dt, \tag{2.0.4}$$

$$U_{m+1}(x) = U_m(x) + \delta_{m+1}(x),$$

here $u_{m,i}$ is the ith element of the approximate vector $U_m$. So at each step, we solve a system of linear integral equations. We also assume that $U^\star$ is a root of the equation (2.0.2) such that $[T(U^\star)]^{-1}$ exists and is a continuous map from $W^d$ to $W^d$. Assume further that $T^{'}(U)$ is locally Lipschitz continuous at $U^\star$,

$$\|T^{'}(U) - T^{'}(V)\| \leq L\|U - V\|, \quad \forall U, V \in N(U^\star)$$

where $N(U^\star)$ is a neighborhood of $U^\star$, and $L > 0$ is a constant. Then by application of local convergence theorem [5,6], there exists a $\delta > 0$ such that if

$$\|U_0 - U^\star\| \le \delta,$$

then the Newton's sequence $\{U_m\}$ is well-defined and converges to $U^\star$. Furthermore, for some constant M we have error bounds

$$\|U_{m+1} - U_\star\| \le M\|U_m - U^\star\|^2 \quad and \quad \|U_m - U^\star\| \le (M\delta)^{2^m}/M.$$

## 3. The Tau's method applied to (2.0.4)

Consider the equation (2.0.4)

$$\delta_{m+1}(x) - \lambda.\int_a^b \kappa(U_m(t))\delta_{m+1}(t)dt = -U_m(x) + F(x) + \lambda.\int_a^b K(x,t,u_{m,1}(t),\dots,u_{m,d}(t))\,dt,$$

$$U_{m+1}(x) = U_m(x) + \delta_{m+1}(x) \quad ,m = 0,1,2,\dots\ .$$

Let $\underline{X} = \{1, x, x^2, x^3, ...\}$ be standard polynomial basis. Now we convert the equation (2.0.4) to the corresponding linear algebraic equations. Let us assume that

$$\frac{\partial K_i(x,t,u_1(t),\dots,u_d(t))}{\partial u_j}|_{U=U_m} = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} K_{ks}^{ij} x^k t^s \quad ,i,j = 1,\dots,d$$

$$\delta_{m+1}(x) = [\ \delta_{m+1,1}(x),\dots,\delta_{m+1,d}(x)\ ]^t = [\ \sum_{j=0}^{\infty} a_{1j}x^j,\dots,\sum_{j=0}^{\infty} a_{dj}x^j\ ]^T = A\underline{X},$$

where

$$A = [a_{ij}] \quad ,i = 1,\dots,d \quad ,j = 0,1,\dots$$

and

$$-U_m(x) + F(x) + \lambda.\int_a^b K(x,t,u_{m,1}(t),\dots,u_{m,d}(t))dt = [\ \sum_{j=0}^{\infty} f_{1j}x^j,\dots,\sum_{j=0}^{\infty} f_{dj}x^j\ ]^T = F\underline{X}$$

with

$$F = [f_{ij}] \quad ,i = 1,\dots,d \quad ,j = 0,1,\dots$$

Then one can writes

$$\int_a^b \frac{\partial K_i(x,t,u_1(t),\dots,u_d(t))}{\partial u_j}|_{U=U_m}\delta_{m+1,j}(t)dt$$
$$= \sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} K_{ks}^{ij} a_{jq} x^k \int_a^b t^s t^q dt$$
$$= \underline{a}_j \underline{K}^{ij} \underline{X},$$

where

$$\underline{a}_j = [a_{j1}, a_{j2}, \dots],$$

$$\underline{K}^{ij} = \begin{pmatrix} \sum_{j=0}^{\infty} K_{0j}\alpha_{j0} & \dots & \sum_{j=0}^{\infty} K_{nj}\alpha_{j0} & \dots \\ \vdots & & \vdots & \\ \sum_{j=0}^{\infty} K_{0j}\alpha_{jn} & \dots & \sum_{j=0}^{\infty} K_{nj}\alpha_{jn} & \dots \\ \vdots & & \vdots & \end{pmatrix}$$

with

$$\alpha_{jl} = \int_a^b t^j t^l dt = \frac{1}{j+l+1}(b^{j+l+1} - a^{j+l+1}), \quad for \;\; j,l = 0,1,...$$

so we have the following form for the integral part of the system

$$\int_a^b \kappa(u^m(t))\delta^{m+1}(t)dt = \begin{pmatrix} \sum_{j=1}^{d} \underline{a}_j \underline{K}^{1j} \underline{X} \\ \vdots \\ \sum_{j=1}^{d} \underline{a}_j \underline{K}^{dj} \underline{X} \end{pmatrix},$$

and the coefficients of exact solution $\delta_{m+1}(x)$ of problem (2.0.4) satisfies the following infinite algebraic system

$$\mathbf{aG} = \mathbf{g}$$

where

$$\mathbf{a} = [\ \underline{a}_1\ \underline{a}_2 \dots\ \underline{a}_d\ ],$$

and $\mathbf{G}$ is a block matrix,

$$\mathbf{G} = \begin{pmatrix} I - \lambda_1 K^{11} & -\lambda_2 K^{21} & \dots & -\lambda_d K^{d1} \\ -\lambda_1 K^{12} & I - \lambda_2 K^{22} & \dots & -\lambda_d K^{d2} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_1 K^{1d} & -\lambda_2 K^{2d} & \dots & I - \lambda_d K^{d1} \end{pmatrix}$$

and

$$\mathbf{g} = \begin{pmatrix} F_1^t \\ F_2^t \\ \vdots \\ F_d^t \end{pmatrix}$$

where $F_i$ denotes the ith row of matrix $F$.

**Definition.** The polynomial

$$(\delta_{m+1})_n = \mathbf{a}_n\ \underline{X}$$

will be called an approximate solution of (2.0.4), if $\mathbf{a}_n = (\underline{a}_{1,n}, \underline{a}_{2,n}, ..., \underline{a}_{d,n})$ , $\underline{a}_{i,n} = [a_{i0}, a_{i1}, \dots, a_{in}]$, is the solution of the system of linear algebraic equations

$$\mathbf{a}_n \mathbf{G}_n = \mathbf{g}_n$$

where elements of $\mathbf{G}_n$ is the restriction of elements of $\mathbf{G}$ to first (n+1) rows and (n+1) columns, similarly for $\mathbf{g}_n$.

## 4. Numerical examples

In this section we apply the above methods to some examples in order to compare numerical solution with analytic solution.

**Example 1.** Consider the following system of nonlinear Fredholm integral equations of second kind

$$\begin{cases} u_1(x) - \int_0^1 (xtu_1^2(t) + t^2u_2^3(t))dt = \frac{-1}{9} + \frac{3}{4}x \\ u_2(x) - \int_0^1 (tu_1^3(t) - xu_2(t))^2 dt = \frac{-1}{9} + \frac{2}{7}x + \frac{4}{5}x^2 \end{cases}$$

with exact solution $u_1(x) = x$ and $u_2(x) = x^2$. The numerical results with initial guess $u_{0,1}(x) = 0$ and $u_{0,2}(x) = 0$ are given in Table1 and Table2 (m: number of iterations in Newton's method, n: degree of polynomial in Tau's method)

Table 1- Numerical results for Example1 and for m=10 and n=2

| x | Computed $u_1(x)$ | Relative Error | Computed $u_2(x)$ | Relative Error |
|---|---|---|---|---|
| 0.0 | -0.00000358 | — | 0.00032421 | — |
| 0.2 | 0.20004537 | 2.2685 e-004 | 0.04024127 | 6.0318 e-003 |
| 0.4 | 0.40009432 | 2.3580 e-004 | 0.16016220 | 1.0138 e-003 |
| 0.6 | 0.60014326 | 2.3877 e-004 | 0.36008701 | 2.4169 e-004 |
| 0.8 | 0.80019221 | 2.4026 e-004 | 0.64001570 | 2.4531 e-005 |
| 1.0 | 1.00024116 | 2.4116 e-004 | 0.99994828 | 5.1720 e-005 |

Table 2- Numerical results for Example1 and for m=15 and n=3

| x | Computed $u_1(x)$ | Relative Error | Computed $u_2(x)$ | Relative Error |
|---|---|---|---|---|
| 0.0 | -0.00000016 | — | 0.00000105 | — |
| 0.2 | 0.19999980 | 1.0000 e-006 | 0.04000080 | 2.0000 e-005 |
| 0.4 | 0.39999976 | 6.0000 e-007 | 0.16000053 | 3.3125 e-006 |
| 0.6 | 0.59999971 | 4.8333 e-007 | 0.36000025 | 6.9444 e-007 |
| 0.8 | 0.79999967 | 4.1250 e-007 | 0.63999995 | 7.8125 e-008 |
| 1.0 | 0.99999963 | 3.7000 e-007 | 0.99999965 | 3.5000 e-007 |

**Example 2.** Consider the following system of nonlinear Fredholm integral equations of second kind

$$\begin{cases} u_1(x) - \int_0^1 e^{-t}(u_1^2(t) - xu_1(t))u_2(t)dt = \frac{2}{3} - \frac{1}{2}x \\ u_2(x) - \int_0^1 (e^{x-2t}u_2^2(t) + xu_1^3(t))dt = \frac{-1}{4}x \end{cases}$$

with exact solution $u_1(x) = 1 - x$ and $u_2(x) = e^x$. The numerical results with initial guess $u_{0,1}(x) = 1$ and $u_{0,2}(x) = 1$ are given in Table3 and Table4, (m: number of iterations in Newton's method, n: degree of polynomial in Tau's method)

Table 3- Numerical results for Example2 and for m=6 and n=6

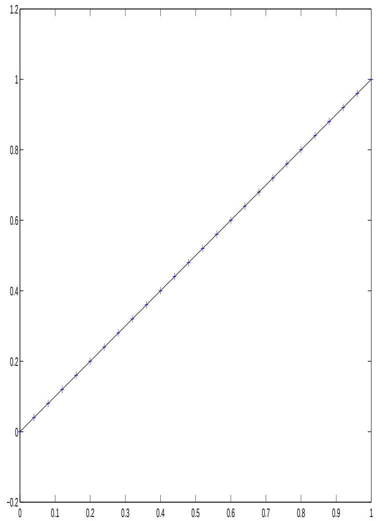

FIGURE 1. (Table1, $u_1(x)$)

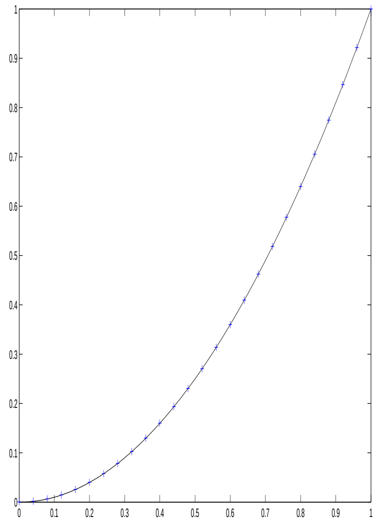

FIGURE 2. (Table1, $u_2(x)$)

| x | Computed $u_1(x)$ | Relative Error | Computed $u_2(x)$ | Relative Error |
|---|---|---|---|---|
| 0.0 | 0.99999061 | 9.3900 e-006 | 1.00006759 | 6.7590 e-005 |
| 0.2 | 0.79998426 | 1.9675 e-005 | 1.22147643 | 6.0316 e-005 |
| 0.4 | 0.59997791 | 3.6817 e-005 | 1.49190697 | 5.5147 e-005 |
| 0.6 | 0.39997156 | 7.1100 e-005 | 1.82220821 | 4.9069 e-005 |
| 0.8 | 0.19996521 | 1.7395 e-004 | 2.22560760 | 2.9957 e-005 |
| 1.0 | -0.00004114 | — | 2.71819091 | 3.3448 e-005 |

Table 4- Numerical results for Example2 and for m=10 and n=10

| x | Computed $u_1(x)$ | Relative Error | Computed $u_2(x)$ | Relative Error |
|---|---|---|---|---|
| 0.0 | 1.00000000 | 0 | 1.00000000 | 0 |
| 0.2 | 0.80000000 | 0 | 1.22140275 | 0 |
| 0.4 | 0.60000000 | 0 | 1.49182469 | 0 |
| 0.6 | 0.40000000 | 0 | 1.82211880 | 0 |
| 0.8 | 0.20000000 | 0 | 2.22554092 | 0 |
| 1.0 | 0.00000000 | — | 2.71828180 | 7.3576 e-009 |

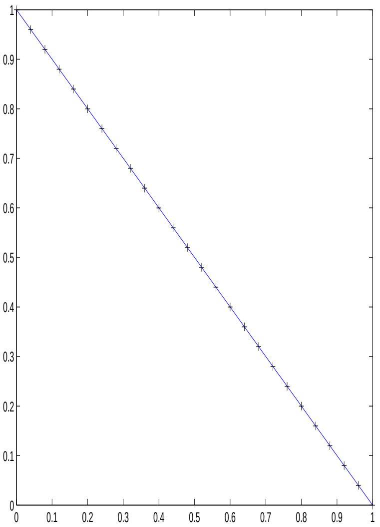


FIGURE 3. (Table3, $u_1(x)$)

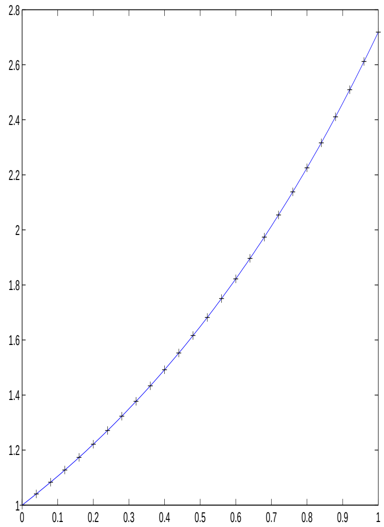


FIGURE 4. (Table3, $u_2(x)$)

## 5. FINAL REMARKS

We see that if the number of iterations in the Newton's method and Tau's method increase, the computed solutions converges to exact solutions. All computations were carried out using MATLAB 7 on a PC.
In future work, we use this method for numerical solution of a system of nonlinear integro-differential equations.

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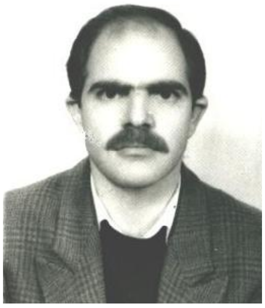

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