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ABOUT USE OF THE BASS RELATIONS FOR SOLUTION OF MATRIX EQUATIONS

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Abstract. The possibility of solution of some matrix equations by the help of Bass relation and its generalizations is shown. It is demonstrated, that the Bass relation "works" even when the standard procedures become inefficient. Comparison of accuracy of the solutions obtained with the help of Bass relation and other methods is carried out on the examples.

Keywords: nonlinear algebraic equations, Bass relation, Hamilton matrix, matrix polynomials, Sylvester equation, characteristic polynomial, factorization, symplectic pencil.

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Introduction

Various algorithms (for further references, please, see for example [6, 13,18, 20,25]) have been offered for the solutions of matrix equations. The use of matrix polynomials for finding solutions of the linear and nonlinear matrix equations has also drawn the attention of researchers for a long time (see, for example [10] where further references may be found). We shall note the approach connected to the Bass relation [15] (p.251) which allows one to find the stabilizing solution of the algebraic Riccati equation

$$A'X + XA - XDX + Q = 0. \tag{1}$$

Hereinafter, the prime means transposing. The essence of this approach consists of the following. To equation (1) there corresponds the Hamiltonian matrix H

$$H = \begin{bmatrix} A & -D \\ -Q & -A' \end{bmatrix}. \tag{2}$$

The characteristic polynomial

$$P(\lambda) = \det(I\lambda - H) \tag{3}$$

has roots symmetrically located with respect to the imaginary axis. This polynomial is factorized as

$$P(\lambda) = \varphi(\lambda)\ \varphi(-\lambda),$$

where the roots of the polynomial $\varphi(\lambda)$ lie in the left half plane. The stabilizing solution of (1) satisfies the following Bass relation [15]

$$\varphi(H) \begin{bmatrix} I \\ X \end{bmatrix} = 0. \tag{4}$$

Hereinafter I is the unit matrix of corresponding size.

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Similar relations have been proposed in [2, 3, 21] for finding the solution of the discrete algebraic Riccati equation (DARE)

$$X = \psi' \left[X - X\Gamma(C + \Gamma' X\Gamma)^{-1}\Gamma' X \right] \psi + Q. \tag{5}$$

Assuming invertibility of the matrices ψ, C, we assign to equation (5) the matrix

$$H_d = \begin{bmatrix} \psi + \Gamma C^{-1}\Gamma'(\psi')^{-1}Q & -\Gamma C^{-1}\Gamma'(\psi')^{-1} \\ -(\psi')^{-1}Q & (\psi')^{-1} \end{bmatrix}.$$

Zeros of the characteristic polynomial $P_d(\lambda) = \det(I\lambda - H_d)$ of this matrix are located symmetrically with respect to the unit circle, i.e. if λ is a root of this polynomial then $\frac{1}{\lambda}$ is also its root.

We factorize $P_d(\lambda)$ as

$$P_d(\lambda) = \varphi_+(\lambda)\varphi_-(\lambda),$$

in such a way that the zeros of $\varphi_-(\lambda)$ lay inside and those of $\varphi_+(\lambda)$ outside the unit circle. In this case, the analogue of Bass relation indeed is [2, 3]

$$\varphi_-(H_d) \begin{bmatrix} I \\ X \end{bmatrix} = 0. \tag{6}$$

However, as it is noted in [15], simplicity of the analytical structure of relations (4) does not guarantee success at direct numerical realization. One of the reasons for this is the necessity of finding the characteristic polynomial of the matrix H_d. Note, that there is a procedure in package MATLAB for the finding the characteristic polynomial of a matrix or of a matrix beam. But usually, through development of numerical algorithms use of this procedure is avoided.

In this connection, to find the solution of (1)(respectively, (5)) if eigenvalues of the matrix H (*respectively*, H_d) do not lie on the imaginary axis (respectively, do not lie on the unit circle) more effective methods (as well as the method of matrix signum functions [4, 8, 11], the doubling algorithm [17, 22] and etc.) are usually used. But, if the matrix H ((*respectively*, H_d) has eigenvalues on the imaginary axis (respectively, on the unit circle), the above mentioned methods become inefficient (additional investigation of the speed of convergence [7] or corresponding modification [5] are required).

In such a situation there can be effective use of the Bass relations (4)and (6). In this case it is expedient to modify this relation removing the requirement of invertibility of the matrices ψ, C in (5). The essence of this approach will be illustrated below by an example of a linear equation (generalized Sylvester equation [26])

$$EX - AXB = C. \tag{7}$$

Further this approach will be used for finding the solution of the nonlinear equations [1, 9, 12, 14, 19, 22-24]

$$X + A'X^{-1}A = Q, \tag{8}$$

$$X - A'X^{-1}A = Q. \tag{9}$$

The comparison of the accuracy of equations (8), (9), obtained by the use of Bass relation and other methods, is illustrated by other authors' examples. On example 3, in which the condition of the theorem 1 [23] on the existence of positively defined symmetric solution of (8) is not satisfied, with the help of Bass relation the non-symmetric solutions are found. In Example 4, in which matrices A and Q are singular, the solution of the equation (9) is constructed. It must to be noted that in this situation the use of the algorithms of [9, 22, 23] is problematic.

1. Linear equation

Let us rewrite equation (7) in the following form

$$M_1 \begin{bmatrix} I \\ X \end{bmatrix} = F_1 \begin{bmatrix} I \\ X \end{bmatrix} B, \tag{10}$$

$$M_1 = \begin{bmatrix} B & 0 \\ -C & E \end{bmatrix}, F_1 = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}.$$

The problem is then construction of the procedure which allows one to transform (10) to the form

$$M_p \begin{bmatrix} I \\ X \end{bmatrix} = F_p \begin{bmatrix} I \\ X \end{bmatrix} \Pi(B), \tag{11}$$

where $\Pi(B)$ is some polynomial of the matrix B. It is obvious, that if $\Pi(B) = 0$ (for example, $\Pi(B)$ is a characteristic polynomial of the matrix B), then the relation (11) turns to the following linear equation (analogue of (4), (6)) in X

$$M_p \begin{bmatrix} I \\ X \end{bmatrix} = 0. \tag{12}$$

Splitting the matrix M_p into blocks $M_p = \begin{bmatrix} M_{p1} & M_{p2} \end{bmatrix}$ (12) can be rewritten as

$$M_{p2}X = -M_{p1}.$$

It is obvious, that (12) defines the solution X only if the matrix M_{p2} is of full rank. In other words, the proposed algorithm "works" only for this case.

Let us note, that if the matrix F_1 is invertible, then by transforming (10) to the form

$$H_f \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} B, H_f = F_1^{-1} M_1 \tag{13}$$

it is easy to get (12), where $M_p = \Pi(H_f)$. However, if the matrix F_1 is singular, then we offer the following approach.

Let A, E be singular matrices. Then to transform (10) to the form (11), we define the matrices M_2, F_2 in the relation

$$M_2 \begin{bmatrix} I \\ X \end{bmatrix} = F_2 \begin{bmatrix} I \\ X \end{bmatrix} B^2. \tag{14}$$

Multiplying (10) from right by B we get

$$M_1 \begin{bmatrix} I \\ X \end{bmatrix} B = F_1 \begin{bmatrix} I \\ X \end{bmatrix} B^2. \tag{15}$$

Let us introduce matrices G_1, L_1 satisfying the relation

$$L_1 M_1 = G_1 F_1. \tag{16}$$

Multiplying equations (10), (15) from left by L_1 and G_1 we get

$$G_1 M_1 \begin{bmatrix} I \\ X \end{bmatrix} = L_1 F_1 \begin{bmatrix} I \\ X \end{bmatrix} B^2. \tag{17}$$

Thus, instead of the M_2, F_2 appearing in (14), one may take the following ones

$$M_2 = G_1 M_1, F_2 = L_1 F_1. \tag{18}$$

According to (16), the matrix $[L_1\ \ G_1]'$ -, is in the kernel of the matrix $\begin{bmatrix} M_1 \\ -F_1 \end{bmatrix}'$. Therefore the matrices L_1, G_1 may be found, as in [17], using procedure null.m package MATLAB. For this purpose, it is possible to use the analytical expressions for the matrices $G_1,\ L_1$ given in [16].

A similar procedure can be used also for construction of the relation with higher degrees of the matrix B.

Let (10) be transformed to the form

$$M_k \begin{bmatrix} I \\ X \end{bmatrix} = F_k \begin{bmatrix} I \\ X \end{bmatrix} B^k. \tag{19}$$

Let us construct a similar relation in which the matrix B has degree $k+1$. We multiply on (19) the right by B and introduce the matrices $G_k,\ L_k$ satisfying

$$L_k M_k = G_k F_1. \tag{20}$$

Then we have,

$$G_k M_1 \begin{bmatrix} I \\ X \end{bmatrix} = L_k F_k \begin{bmatrix} I \\ X \end{bmatrix} B^{k+1}.$$

Thus $M_{k+1} = G_k M_1$, $F_{k+1} = L_k F_k$.

According to (20), the matrix $\begin{bmatrix} L_k & G_k \end{bmatrix}'$ is in the kernel of the matrix $\begin{bmatrix} M_k \\ -F_1 \end{bmatrix}'$ and therefore, as it was already noted, to find the matrices $G_k,\ L_k$ it is possible to use the procedure null.m package MATLAB.

We note, that as $F_{k+1} = L_k F_k$,

$$F_k = L_{k-1} \dots L_1 F_1. \tag{21}$$

Thus, we described above the procedure allowing one to construct the matrices $M_k,\ F_k$ appearing in (19). Let us use this procedure to define M_p in (12). We assume, that the polynomial appearing in (11)

$$\Pi(B) = \beta_0 B^m + \beta_1 B^{m-1} + \dots + \beta_{m-1} B + \beta_m I \tag{22}$$

is a characteristic polynomial of the matrix B, with $\beta_0 = 1$. For this purpose, as the first step, it is necessary to take the same matrix coefficients at B^k for $k = 1, \dots, m$. To do this taking into account (21), we multiply relations (19) from left by $L_m, L_{m-1}, \dots, L_k$, correspondingly. We add here the identity

$$F_m \begin{bmatrix} I \\ X \end{bmatrix} = F_m \begin{bmatrix} I \\ X \end{bmatrix}. \tag{23}$$

Let us multiply (23) by β_m, and relations where B^k appears by β_{m-k} (the coefficients β_i are defined by (22)). Adding them, we obtain

$$(\beta_m F_m + \beta_{m-1} L_m L_{m-1} \dots L_1 M_1 + \dots + \beta_1 L_m M_{m-1} + M_m) \begin{bmatrix} I \\ X \end{bmatrix} = F_m \begin{bmatrix} I \\ X \end{bmatrix} \Pi(B) = 0.$$

Hence, the matrix M_p appearing in (12) indeed is

$$M_p = \beta_m F_m + \beta_{m-1} L_m L_{m-1} \dots L_1 M_1 + \dots + \beta_1 L_m M_{m-1} + M_m. \tag{24}$$

Thus, the solution of the equation (7) is defined by (12) in which the matrix M_p looks as (24). Coefficients β_i are defined by (22), and matrices M_1, F_1 by (10).

Example 1. Let us compare the accuracy of the preceding algorithm with the accuracy of standard procedure of package MATLAB. Let in (7)

$$A=\frac{1}{6}\begin{bmatrix}3&0&0&0\\3&4&1&1\\1&1&3&4\\2&1&0&3\end{bmatrix},B=\begin{bmatrix}1&1\\1&1-\varepsilon\end{bmatrix},\varepsilon=10^{-6},X_0=\begin{bmatrix}1&2\\3&4\\5&6\\7&8\end{bmatrix},$$

$$E=diag\left\{1\ \ 1\ \ 0\ \ 1\right\},C=EX_0-AX_0B.$$

Here the matrix A is invertible, therefore, transforming equation (7) to the form

$$A^{-1}EX-XB=A^{-1}C, \tag{25}$$

for its solution it is possible to use procedure lyap.m of the MATLAB package. Accuracy of the obtained solution can be characterized as

$$er_M=\|X-X_0\|_\infty=1.9\cdot10^{-7}.$$

As a result of using the procedure described above (based on (12)) the solution of (7) has been obtained with the following size of the error

$$er=\|X-X_0\|_\infty=3.75\cdot10^{-8}.$$

Comparing er_M and er one may ascertain that in the given example accuracy of the proposed algorithm is higher than the accuracy of the standard procedure of the MATLAB package.

2. EQUATION (8).

As it is noted in [22], this equation can be presented similarly to (10)

$$M_1\begin{bmatrix}I\\X\end{bmatrix}=F_1\begin{bmatrix}I\\X\end{bmatrix}B,B=X^{-1}A, \tag{26}$$

$$M_1=\begin{bmatrix}A&0\\Q&-I\end{bmatrix},F_1=\begin{bmatrix}0&I\\A'&0\end{bmatrix}.$$

Thus, the solution of this equation is defined by the eigenvalues and eigenvectors of the matrix pencil

$$M_1-\mu F_1. \tag{27}$$

Let us note, that this pencil is symplectic. Indeed [8]

$$M_1JM_1'=F_1JF_1'=\begin{bmatrix}0&-A\\A'&0\end{bmatrix},J=\begin{bmatrix}0&I\\-I&0\end{bmatrix}.$$

Hence, eigenvalues of the pencil (27) are located symmetrically relative to the unit circle. It in particular follows from the fact that (as it has been noted in [23]) the equation (8) is a special case of the generalized DARE (5)

$$X=\psi'X\psi-(\Gamma'X\psi+A)'(C+\Gamma'X\Gamma)^{-1}(\Gamma'X\psi+A)+Q \tag{28}$$

if $\psi=0,\ C=0,\ \Gamma=I$ in (28).

First we consider the case when the matrix A is invertible and offer a procedure of finding the solution to (8). This assumption allows one to write (26) similarly to (13)

$$H_f\begin{bmatrix}I\\X\end{bmatrix}=\begin{bmatrix}I\\X\end{bmatrix}B,H_f=F_1^{-1}M_1. \tag{29}$$

Let us note, that to each set of n elements from $2n$ eigenvalues of the pencil (27) or matrices H_f in (29) there corresponds some solution of (8) defining the matrix B. It means that depending on a choice of the solution of (8) the spectrum of the matrix B in (26), (29) will coincide with n roots of the characteristic polynomial $\varphi_f(\lambda)$ of the matrix H_f. So, for example, let's factorize $\varphi_f(\lambda)$, i.e. represent it as product of two polynomials of degree n

$$\varphi_f(\lambda) = \varphi_{f-}(\lambda)\varphi_{f+}(\lambda). \tag{30}$$

Here the modules of roots $\varphi_{f+}(\lambda)$ are greater than or equal to 1 and modules of roots $\varphi_{f-}(\lambda)$ are less than or equal to unit. In this case, for some solution X, $\varphi_{f-}(\lambda)$ will be a characteristic polynomial of the matrix B. To find the solution of (8) it is possible to use this fact. We note that according to (29)

$$H_f^k \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} B^k, k = 1, 2, \ldots. \tag{31}$$

Multiplying each relation (31) by the corresponding coefficient $\varphi_{f-}(\lambda)$, combining them and taking into account, that $\varphi_{f-}(B) = 0$, we obtain

$$\varphi_{f-}(H_f) \begin{bmatrix} I \\ X \end{bmatrix} = 0. \tag{32}$$

Similar relations may be obtained also for the other solutions of (8), choosing another factorization of the polynomial $\varphi_f(\lambda)$.

In the case of singular matrices A, to find the solution of (8) it is possible to use the procedure described in section 1. Indeed, equation (26) looks like (10). Thus, it is essential, that in the procedure described in item 1 the concrete form of matrix B is not used, but only its spectrum is used. As it has been noted, this spectrum can be found as a result of the factorization of the characteristic polynomial $\varphi_f(\lambda)$ of the matrix pencil (27)

$$\varphi_f(\lambda) = \det(M_1 - \lambda F_1). \tag{33}$$

So, for example, as a result of factorization (30), the characteristic polynomial $\varphi_{f-}(\lambda)$ of the matrix B appearing in (26) is obtained. Therefore in the case of singular matrix A, the relation (12) may be used for finding the solution of (8). Here the matrix M_p is defined by (24), the matrices M_1, F_1 by (26) and coefficients β_i are defined by the polynomial obtained in the result of factorization of (33).

We note, that in the case of singular matrix A, the pencil (27) will have zero roots and as a result of its symplecticity, the same number of roots on infinity. In this connection, the degree of the polynomial $\varphi_f(\lambda)$ will be less than $2n$. Hence, if in (30) the degree of the polynomial $\varphi_{f-}(\lambda)$ is equal to n, then the degree of $\varphi_{f+}(\lambda)$ will be less than n.

It is natural, that the algorithm described in this example can also be used for finding the solution of DARE (28). In this case, as the matrices M_1 and F_1 in (26) it is possible to take the following matrices [17]

$$M_1 = \begin{bmatrix} \psi - \Gamma W(\Gamma'\psi + A) & 0 \\ A' W(\Gamma'\psi + A) - Q & I \end{bmatrix},$$

$$F_1 = \begin{bmatrix} I - \Gamma W \Gamma' & \Gamma W \Gamma' \\ A' W \Gamma' & \psi' - A' W \Gamma' \end{bmatrix},$$

$$W = (C + \Gamma'\Gamma)^{-1}.$$

Example 2 (Example 3 from [9]). Let the matrices A, Q in (8) be

$$A = \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.15 & 0.15 \\ 0.1 & 0.15 & 0.25 \end{bmatrix}, Q = I.$$

In spite of the fact that equation (8) is a special case of DARE (28), using the standard procedure dare.m package MATLAB does not allow one to find the solution of this example. The matter is that in the given example the characteristic polynomial (33) has roots equal to one. The use of the relation (32) allows one to find the solution X_b of this equation (to find the characteristic polynomial (33) procedure poly.m package MATLAB was used). To this solution there corresponds the following norm of error

$$er_b = \left\| X_b + A' X_b^{-1} A - Q \right\|_\infty = 4.4 \cdot 10^{-13}.$$

It needs to be noted that the solution X_b of equation (8) is maximal [19]. The corresponding minimal solution can be obtained by replacing in (32) $\varphi_{f-}(H_f)$ by $\varphi_{f+}(H_f)$. Using (12) it has been obtained (also maximal) the solution X_p to which there corresponds to the following norm of the error

$$er_p = \left\| X_p + A' X_p^{-1} A - Q \right\|_\infty = 5.5 \cdot 10^{-16}.$$

It needs to be noted that in [9], using the iterative procedures, the solution of (8) has been obtained with the norm of error of the order 10^{-8}. Using linear matrix inequalities and iterative procedures, in [19] the solution of (8) is obtained with error norm of order 10^{-11}. Comparing these quantities with er_b, er_p one may ascertain that in this example the proposed algorithm allows one to get the solution of (8) with the essential greater accuracy.

It is essential that the proposed algorithm allows one to find also asymmetrical solutions of (8). In this connection we consider below an example in which the condition of existence of the symmetric positively defined solution is not satisfied. The essence of this condition consists of the fact that for all λ lying on the unit circle the function $\psi(\lambda)$ satisfies $\psi(\lambda) = A\lambda + Q + \lambda^{-1} A' \geq 0$.

Example 3. Here the matrices in (8) are taken as

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is obvious, that in this example the condition mentioned above is not satisfied since $\det(\psi(\lambda)) = -3$ when $\lambda = i$. The characteristic polynomial (33) corresponding to the matrices A, Q looks like

$$\varphi_f(\lambda) = \lambda^5 - \lambda^3 + \lambda.$$

This polynomial can be presented in the form of (30) taking

$$\varphi_{f-}(\lambda) = \lambda^3 + \sqrt{3}\lambda + \lambda, \varphi_{f+}(\lambda) = \lambda^2 - \sqrt{3}\lambda + 1.$$

We assume, that the matrix polynomial (22) is defined as $\varphi_{f-}(\lambda)$, i.e. the coefficients $\beta_1,\ \beta_2,\ \beta_3$ have the following values

$$\beta_1 = \sqrt{3}, \beta_2 = 1, \beta_3 = 0.$$

According to (24), the matrix M_p from (12) looks like

$$M_p = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.3329 & -0.6364 & 0 & 0.7175 & 0.0299 \\ 0 & 0.6364 & -0.3329 & 0 & -0.0299 & 0.7175 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -0.1760 & 1.2055 & 0 & -0.9560 & -0.7552 \\ 0 & -1.2055 & -0.1760 & 0 & 0.7552 & -0.9560 \end{bmatrix}.$$

Its right block

$$M_{p2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.7175 & 0.0299 \\ 0 & -0.0299 & 0.7175 \\ -1 & 0 & 0 \\ 0 & -0.9560 & -0.7552 \\ 0 & 0.7552 & -0.9560 \end{bmatrix}$$

is a matrix of a full rank and consequently (12) defines the solution X as

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5000 & 0.8660 \\ 0 & -0.8660 & 0.5000 \end{bmatrix}.$$

Here matrix X is not symmetrical. To the obtained value of X there corresponds the error norm $\cong 10^{-16}$. Taking $\varphi_{f-}(\lambda) = \lambda^3 - \sqrt{3}\lambda^2 + \lambda,\ \ \varphi_{f+}(\lambda) = \lambda^2 + \sqrt{3}\lambda + 1$ we get the second asymmetrical solution of (8) that differs from the obtained above one only by transposing.

3. Equation (9).

According to [22], the solution of this equation satisfies (26) in which the matrices M_1, F_1, B are taken as

$$M_1 = \begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix}, F_1 = \begin{bmatrix} 0 & I \\ A' & 0 \end{bmatrix}, B = X^{-1}A. \tag{34}$$

In this case the matrix pencil defined by the matrices (34) as

$$M_1 - \mu F_1 \tag{35}$$

differs from (24) as it is not symplectic. However, it is essential that in this case, its eigenvalues, in some sense, will be "symmetrically" located relative to the unit circle. Indeed let's rewrite the pencil (35) as

$$M_1 - \xi F_i,\ \xi = \frac{\lambda}{i},\ F_i = iF_1,\ i = \sqrt{-1}. \tag{36}$$

It is easy to check that the pencil (36) will be symplectic, i.e., if ξ is an eigenvalue of this pencil, then $\frac{1}{\xi}$ also will be its eigenvalue. Hence, if λ is eigenvalue of the pencil (35) then $-\frac{1}{\lambda}$ also will be eigenvalue . In this sense it is possible to speak about "symmetry" of the eigenvalues of (35) with respect to unit circle.

Thus, procedure of finding the solution of (9) is similar to the one described above. Indeed, if the matrix A is invertible, the matrices M_1, F_1 are defined by (34) and therefore (26) will be transformed to (29). Then (32) may be used for finding X.

We have the similar situation in the case of singular matrix A. In this case the solution of equation (9) is defined by (12).

We shall illustrate in detail this procedure on the following example.

Example 4. The matrices appearing in (9) are taken as

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The characteristic polynomial $\varphi(\lambda)$ of the pencil (35) corresponding to these matrices is

$$\varphi(\lambda) = \det [M_1 \quad - \lambda F_1] = \lambda^3 - \lambda.$$

Let us factorize $\varphi(\lambda)$ as

$$\varphi(\lambda) = \varphi_-(\lambda)\varphi_+(\lambda),$$

$$\varphi_-(\lambda) = \lambda^2 - \lambda, \ \varphi_+(\lambda) = \lambda + 1. \tag{37}$$

We assume that the matrix polynomial (22) is defined as $\varphi_-(\lambda)$, i.e. the coefficients β_1, β_2 are

$$\beta_1 = -1, \ \beta_2 = 0.$$

Accordingly to (24) the matrix M_p appearing in (12) is found as

$$M_p = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ -1 & 0 & 1 & 0 \\ 0 & -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}.$$

Its right block,

$$M_{p2} = \begin{bmatrix} 0 & 0 \\ 0 & -\sqrt{2}/2 \\ 1 & 0 \\ 0 & \sqrt{2}/2 \end{bmatrix}$$

is a matrix of full rank.

These matrices define the following solution of (9)

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If take $\varphi_-(\lambda) = \lambda^2 + 1, \ \ \varphi_+(\lambda) = \lambda - 1$ in (37), i.e.

$$\beta_1 = 1, .\beta_2 = 0,$$

then according to (12) we get one more solution of (9)

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let us note that in connection with the singularity of the matrices A, Q the solution of this example cannot be obtained with the help of the algorithms of [9, 22, 23].

Example 5 (Example 7 from [9]). The matrices appearing in (9) are taken as

$$A = \begin{bmatrix} 50 & 20 \\ 10 & 60 \end{bmatrix} Q = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}.$$

The use of the relation (32) allows one to find solution X_b of equation (9). To this solution there corresponds the following error norm

$$er_b = \left\| X_b - A' X_b^{-1} A - Q \right\|_\infty = 5.68 \cdot 10^{-14}.$$

The relation (12) allows one to get the solution X_p of the equation (9) with error norm

$$er_p = \left\| X_p - A' X_p^{-1} A - Q \right\|_\infty = 1.39 \cdot 10^{-13}.$$

In [9], by using various iterative procedures, the solution of this example has been obtained, error norms of which have orders 10^{-10}, 10^{-11}. Comparing these quantities with er_b, er_p, we can ascertain that the in this example the proposed algorithm allows one to get the solution of (9) with essentially better accuracy.

4. Conclusions

It is shown that by the help of Bass relation and its generalizations it is possible to find solutions of the various matrix equations. It is noted that Bass relation "works" even when standard procedures become inefficient. Comparison of accuracy of the solutions obtained with the help of Bass relation and other methods is carried out on examples.

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